# Modelling Dependence for Credit Derivatives with Copulae*

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*The original paper may be downloaded from http://gro.creditlyonnais.fr.

# Agenda

- Some Examples of Popular Credit Derivatives
- The one-credit intensity framework
- Copulae in a nutshell
- How to introduce dependence in credit intensity models: a copula approach
- Discussion of the model

# 1 Some Examples Of Popular Credit Derivatives

Two major categories of credit derivatives:

- claims that are contingent to an event of default;
- claims that are contingent to a change of rating or spread.

We are only concerned with default-linked credit derivatives, and even ‘pure’ credit products (there exists hybrid interest rate - credit derivatives).

## 1.1 Credit Default Swap

This the plain-vanilla credit derivative product.

A Credit Default Swap contract offers protection against the default of a certain underlying entity within a specified time horizon.

Like an interest-rate swap, it consists of two legs:

- On the 'fixed' leg the holder pays a premium (*spread*) on a regular basis until the maturity or default of the underlying credit.
- On the 'default' leg the holder receives in case of default the loss suffered on the underlying entity (times $1 - R$ where $R$ is the recovery rate).

## 1.2 Nth-To-Default Contract

This is the simplest basket credit derivative.

This product is like a classical Credit Default Swap but it is written on a basket of credits (usually on 5-10 names).

- On the 'fixed' leg the holder pays a premium (*margin*) on a regular basis until the maturity or occurrence of the $N^{\text{th}}$ default.
- On the 'default' leg the holder receives in case of the $N^{\text{th}}$ default the loss suffered on the corresponding defaulting entity (times $1 - R_i$ where $R_i$ is the corresponding recovery rate).

## 1.3 Collateralized Debt Obligation

This product gives a protection against a certain amount of loans in a credit portfolio ($>$ 100 names).

We consider a portfolio of $I$ credits with respective notionals $N_i$, recovery rates $R_i$, maturities $T_i$ and spreads $s_i(T)$. We define a 'tranching' of a product that is a partition $(N_\alpha)$ of the total notional $\sum_i N_i$. The aggregate loss at time $t$ is defined by:

$$L(t) = \sum_{\tau_i \leq t \wedge T_i} (1 - R_i) N_i.$$

Then the holder of the tranche $\alpha$ of the CDO has a guarantee against the loss in the tranche $N_\alpha$.

Example: we consider three tranches :

| nb | name | value |
|---|---|---|
| 1 | junior | 5 % |
| 2 | mezzanine | 30 % |
| 3 | senior | 65 % |

The holder of the mezzanine tranche will be protected against the default between 5 % and 30 % of the aggregate notional.

- On the ‘fixed’ leg the holder pays a premium on a regular basis until maturity of the product.
- On the default leg the holder receives on the same basis the loss suffered on the period that belongs to the tranche.

# 2 The Intensity Framework

This is an alternative to Merton's structural model. In Merton's model the default occurs when the stock price of the firm falls below a pre-specified deterministic threshold (debt of the firm). But the default time is then **predictible**.

Characteristics of the intensity model (Duffie, Lando):

- the Intensity model allows to add some randomness to the default threshold, in such a way that the default occurs as a complete surprise.
- this model loses the micro-economic interpretation of the default time (the model comes from reliability theory), but traders do not care for the purpose of pricing.

## 2.1 Construction Of The Default Time

The default time of a firm is often defined by

$$\tau_1 := \inf\left\{t : \int_0^t \lambda_s^1 \, ds \geq \theta_1\right\}, \qquad \theta_1 \perp\!\!\!\perp (\lambda_t^1, r_t, \ t \geq 0)$$

- $\lambda^1$ a nonnegative, continuous, adapted process called the **intensity process**. It contains the information on the credit quality of firm 1. Here, for simplicity, we will suppose it to be deterministic in the pricing examples.
- $\theta_1$ is a random threshold (usually an exponential r.v. of parameter 1), independent of the intensity.
- we do not model any recovery rate here i.e. $R_1$ is deterministic.

## 2.2 Pricing Default Zero-Coupons

The independence of $\theta_1$ and the intensity process allows to price all derivatives (closed formulae or Monte Carlo simulation). The defaultable zero-coupon of firm 1 is given by (as long as the firm has not defaulted)

$$B_1(t,T) = \mathbb{E}\left[e^{-\int_t^T (r_s + \lambda_s^1)\,\mathrm{d}s} \middle| \mathcal{F}_t\right]$$

When interest rates and intensities are deterministic we have:

$$B_1(t,T) = B_0(t,T)\frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}.$$

**When there is only one credit, we can identify the intensity process as the spread of the firm.**

## 2.3 Calibration Of The Intensity

The (deterministic) intensity is calibrated on Credit Default Swaps market prices. We give the formula for the price of a Credit Default Swap of maturity $T$:

$$\begin{aligned}\mathrm{CDS}_1(T) = \quad & (1-R_1)\int_0^T B_0(0,u)\mathbb{P}(\tau_1 \in \mathrm{d}u) \\ & - s_1(T)\sum_i \Delta T_i B_0(0,T_i)\mathbb{P}(\tau_1 \geq T_i)\end{aligned}$$

The calibration is easy when we choose piecewise constant or affine intensity function.

In general, when the term structure is flat $s_1(T) = s_1$, a good approximation of the intensity is given by:

$$\lambda_1 = \frac{s_1}{1-R_1}$$

## 2.4 Multi-Credit Extensions

When dealing with more than one firm, there are many ways to incorporate dependence in the model. Example with two firms:

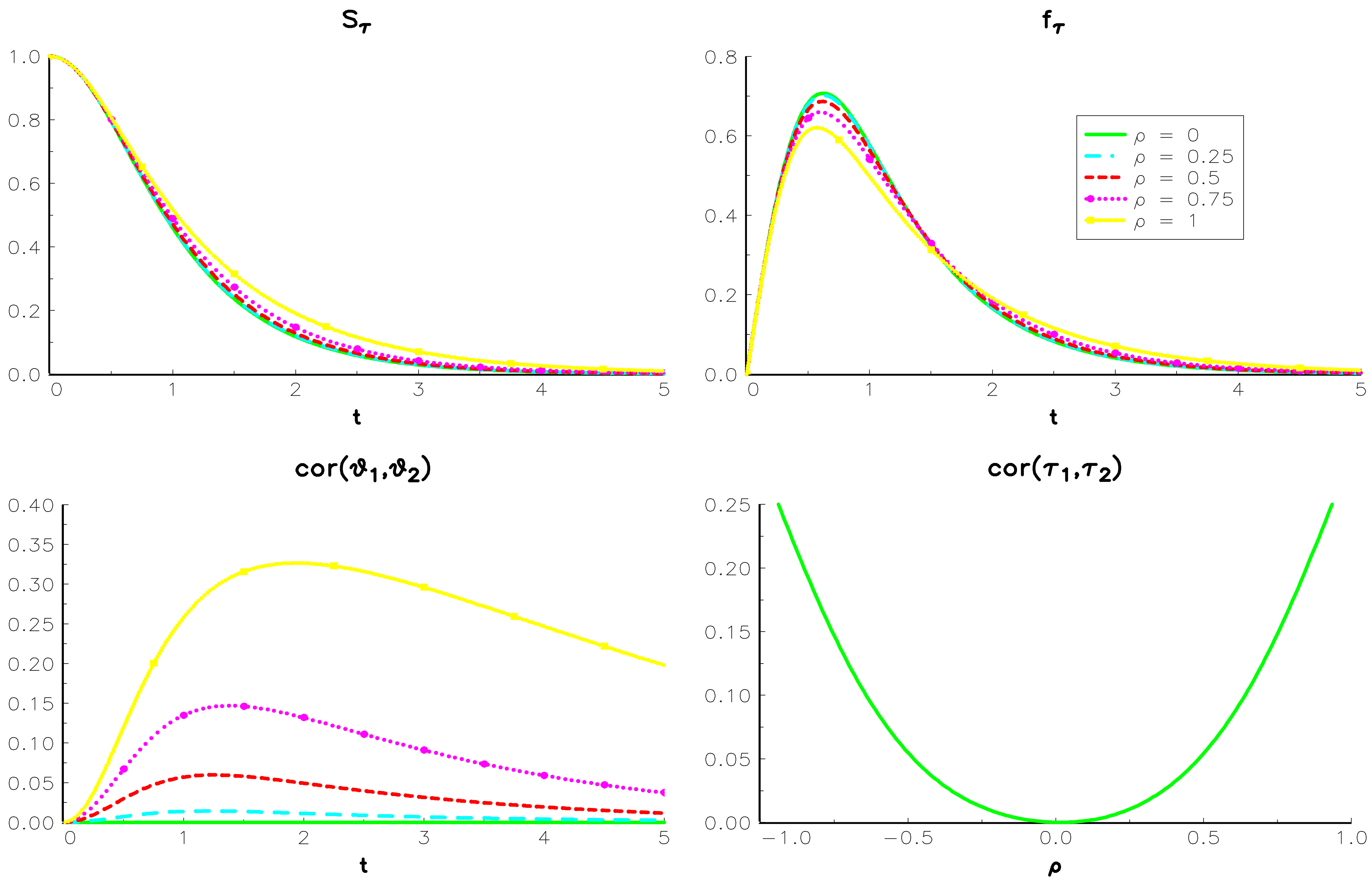
$$\begin{array}{ccc} \theta_1 & \overset{\perp\!\!\!\perp}{\longrightarrow} & \lambda_t^1, r_t \\ {\scriptstyle (2)}\downarrow & & \downarrow{\scriptstyle (1)} \\ \theta_2 & \underset{\perp\!\!\!\perp}{\longrightarrow} & \lambda_t^2, r_t \end{array}$$

- (1) correlating the intensity (stochastic) processes, but this method provides low correlations between the default times,
- (2) correlating the random thresholds with a survival copula $\breve{C}^{\theta}$ (Schönbucher and Schubert's approach, 2001),
- (3) a more intricate way: $\lambda_2$ may be correlated with $\theta_1$ (Jarrow and Yu, 2001).

Example of (1): correlating the intensity processes.

- We choose for the intensities two Cox-Ingersoll-Ross processes driven by correlated Brownian motions (on the graphics, we choose two squared Brownian motions for simplicity's sake).
- We also draw two independent random thresholds.

When the correlation parameter $\rho$ ranges from -1 to +1 the output correlation between default times is less than 25 %.




Influence of the correlation parameter on the first default time

# 3 Copulae In A Nutshell

**Definition**: a **copula** is the joint probability of any two-dimensional uniform r.v. $(U_1, U_2)$,

$$\mathbf{C}^U(u_1, u_2) := \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2).$$

A copula is thus increasing in its 2 arguments and we have the equalities $\mathbf{C}^U(0, u) = \mathbf{C}^U(u, 0) = 0$ and $\mathbf{C}^U(u, 1) = \mathbf{C}^U(1, u) = u$.

## 3.1 Examples

- The Independent Copula: $\mathbf{C}^{\perp}(u_1, u_2) = u_1 u_2$.
- The Fréchet Bounds Copulae

$$\mathbf{C}^{+}(u_1, u_2) = \min(u_1, u_2) \quad \mathbf{C}^{-}(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$$

We have an inequality which generalizes $-1 \leq \rho \leq +1$ (where $\rho$ is the linear correlation) to copulae:

$$\mathbf{C}^{-}(u_1, u_2) \leq \mathbf{C}(u_1, u_2) \leq \mathbf{C}^{+}(u_1, u_2)$$

## 3.2 Sklar's Representation Lemma

**Key idea**: Copulae are used to split the margins and the dependence of the joint distribution.

**Notation**: for any two-dimensional random variable $X = (X_1, X_2)$, we denote for the marginal and joint survival probabilities:

$$\begin{cases} S^X(x_1, x_2) := \mathbb{P}(X_1 > x_1, X_2 > x_2), \\ S_1^X(x_1) := \mathbb{P}(X_1 > x_1), \quad S_2^X(x_2) := \mathbb{P}(X_2 > x_2). \end{cases}$$

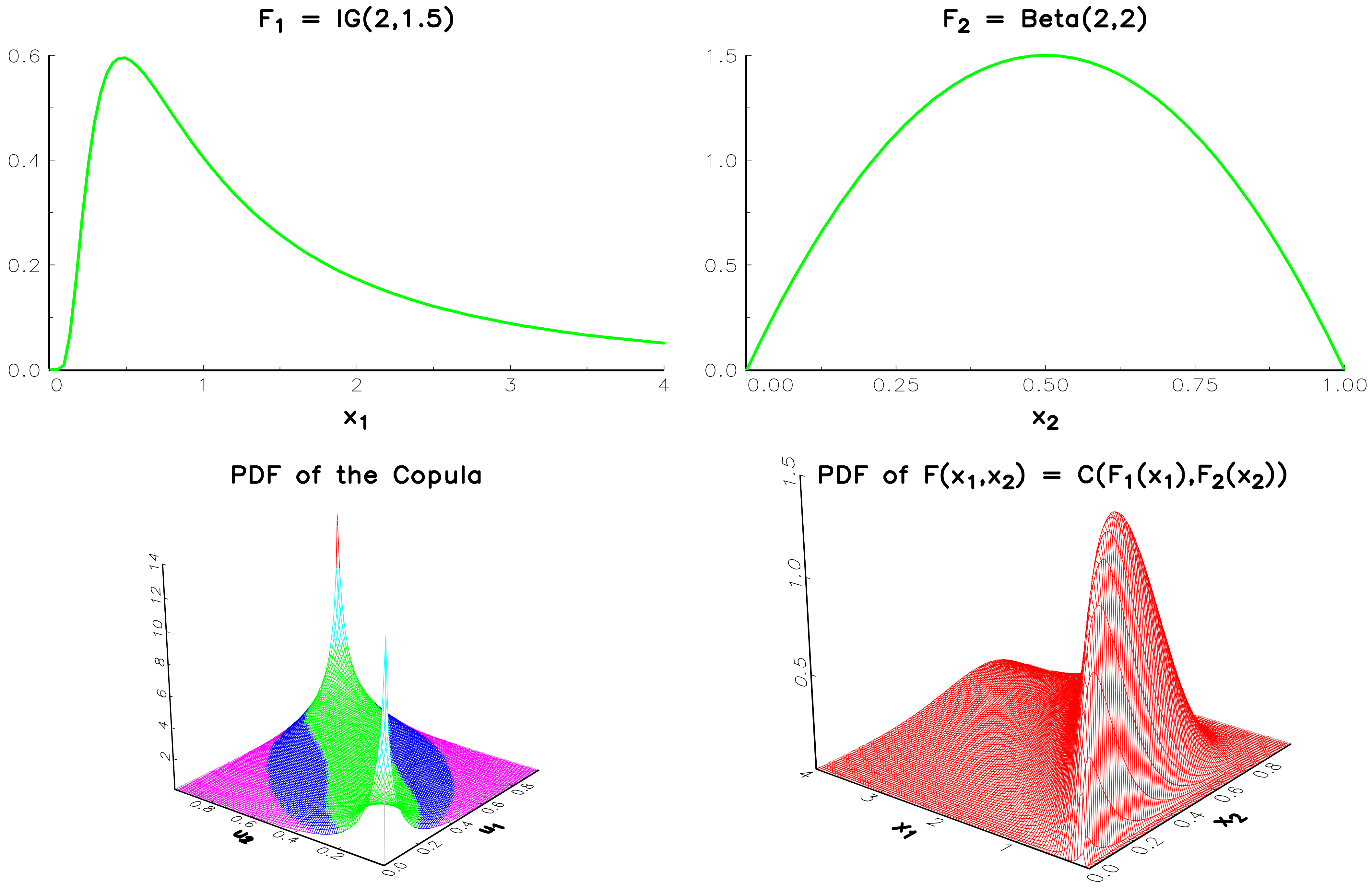
As $S_1^X(X_1)$ and $S_2^X(X_2)$ are **uniform** variables, they admit a copula, which we call the *survival copula* of $X$ and write $\breve{\mathrm{C}}^X$. And we get **Sklar's lemma**:

$$S^X(x_1, x_2) = \breve{\mathrm{C}}^{\mathbf{X}}(S_1^X(x_1), S_2^X(x_2))$$

How to use this result ?

- We can extract copulae from well known bi-variate distributions (e.g. the Gaussian, Student, Gumbel copula families).
- We can create new bi-variate distributions by joining arbitrary margins together with copulae.

All of this can be generalized to multi-variate distributions of higher dimension.




Bivariate distribution with given marginals

# 4 A Copula Multi-Default Model

Common philosophy of all copula models for credit risk:

- Provide a 'smooth' extension of the single-default intensity framework.
- Split the calibration of the spreads and the dependence.

Common shortcomings:

- One must choose an arbitrary copula family (dependency structure of the default times).
- The models needs re-calibrating every day.

## 4.1 The Threshold Approach (Gesiecke, Schönbucher and Schubert)

Here, as we announced, we put a copula $\breve{\mathrm{C}}^{\theta}$ directly on the random thresholds $\theta_i$ (and keep the same construction of default times).

One has to be cautious with this modelling:

- Keep in mind that the thresholds $\theta_i$ are not directly observable market variables.
- Do not mix up the threshold copula $\breve{\mathrm{C}}^{\theta}$ and the default copula $\breve{\mathrm{C}}^{\tau}$ (exception: when all spreads are deterministic both copulae are the same).

## 4.2 The Survival Approach (Li)

This is a special case of the threshold model useful when spread rates are deterministic. It is the case that is mostly used in applications.

We define the random default times as if they were independent:

$$\tau_i := \inf \left\{ t \geq 0 : \int_0^t \lambda_s^i \, \mathrm{d}s \geq \theta_i \right\}, \qquad i = 1, 2.$$

Now, using Sklar's lemma, $\mathbf{S}^\tau$ has a copula representation, which allows us to impose the choice of the copula $\breve{\mathbf{C}}^\tau$,

$$S^\tau (t_1, t_2) = \breve{\mathbf{C}}^\tau \left( S_1^\tau (t_1), S_2^\tau (t_2) \right).$$

## 4.3 Pricing Default Zero-Coupons

We can derive a pricing formula for firm 1's zero-coupon of maturity $T$ at time $t$, as long as no firm has defaulted,

$$B_1(t,T) = \mathbb{E}\left[ e^{-\int_t^T r_s \, \mathrm{d}s} \frac{\breve{\mathrm{C}}^{\theta}\left(e^{-\int_0^T \lambda_s^1 \, \mathrm{d}s}, e^{-\int_0^t \lambda_s^2 \, \mathrm{d}s}\right)}{\breve{\mathrm{C}}^{\theta}\left(e^{-\int_0^t \lambda_s^1 \, \mathrm{d}s}, e^{-\int_0^t \lambda_s^2 \, \mathrm{d}s}\right)} \mid \mathcal{F}_t \right]$$

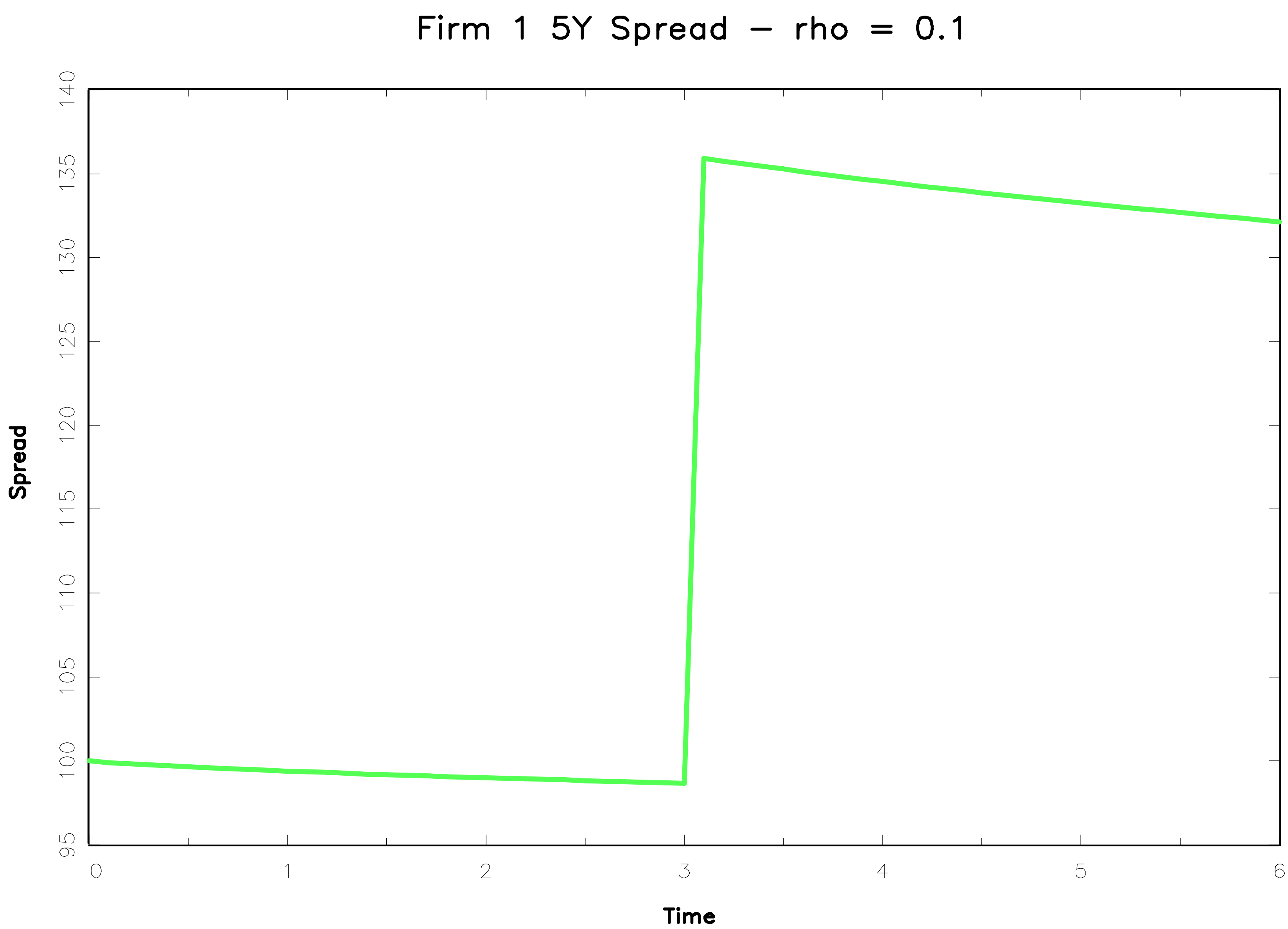
- We notice that firm 2's intensity intervene in the pricing of bond 1's valuation (in particular, default of firm 2 changes firm 1's pricing formula).
- When $\breve{\mathrm{C}}^{\perp}(u_1, u_2) = u_1 u_2$, we retrieve the usual formula.

When firm 2 has defaulted the price of firm 1's zero-coupon becomes (for $t > \tau_2$):

$$B_1(t,T) = \mathbb{E}\left[ e^{-\int_t^T r_s \, \mathsf{d}s} \frac{\partial_2 \breve{\mathrm{C}}^\theta \left( e^{-\int_0^T \lambda_s^1 \, \mathsf{d}s}, e^{-\int_0^{\tau_2} \lambda_s^2 \, \mathsf{d}s} \right)}{\partial_2 \breve{\mathrm{C}}^\theta \left( e^{-\int_0^t \lambda_s^1 \, \mathsf{d}s}, e^{-\int_0^{\tau_2} \lambda_s^2 \, \mathsf{d}s} \right)} \mid \mathcal{F}_t \right]$$

So we observe a jump of the price of zero-coupon of firm 1 when firm 2 defaults, which corresponds to a jump of the spread of firm 1.

**Important: in this model one cannot identify the intensity with the spread at time** $t > 0$.


Firm 1 5Y Spread - rho = 0.1
Spread
Time
95
100
105
110
115
120
125
130
135
140
0
1
2
3
4
5
6

## 4.4 Pricing Nth-To-Default Contracts

We choose a Normal copula and we price first- and Nth-to-default contracts for different values of the (unique) correlation parameter.

We choose two baskets of $I = 4$ credits with the following characteristics ($R = 50\%$). Basket 1 is homogeneous but it is not the case for basket 2.

| credit | basket1 | basket2 |
|---|---|---|
| 1 | 100 bp | 50 bp |
| 2 | 100 bp | 100 bp |
| 3 | 100 bp | 100 bp |
| 4 | 100 bp | 150 bp |

We give here some approximation formulae for the margin of the Nth to default (N = 1...4). We note $s_1, \ldots, s_4$ the spreads of the firms and $m_1, \ldots, m_4$ the fair margins of the first-, ..., fourth-to-default contract.

In case of the independent copula, $\breve{\mathbf{C}}^\tau = \breve{\mathbf{C}}^\perp$, we have:

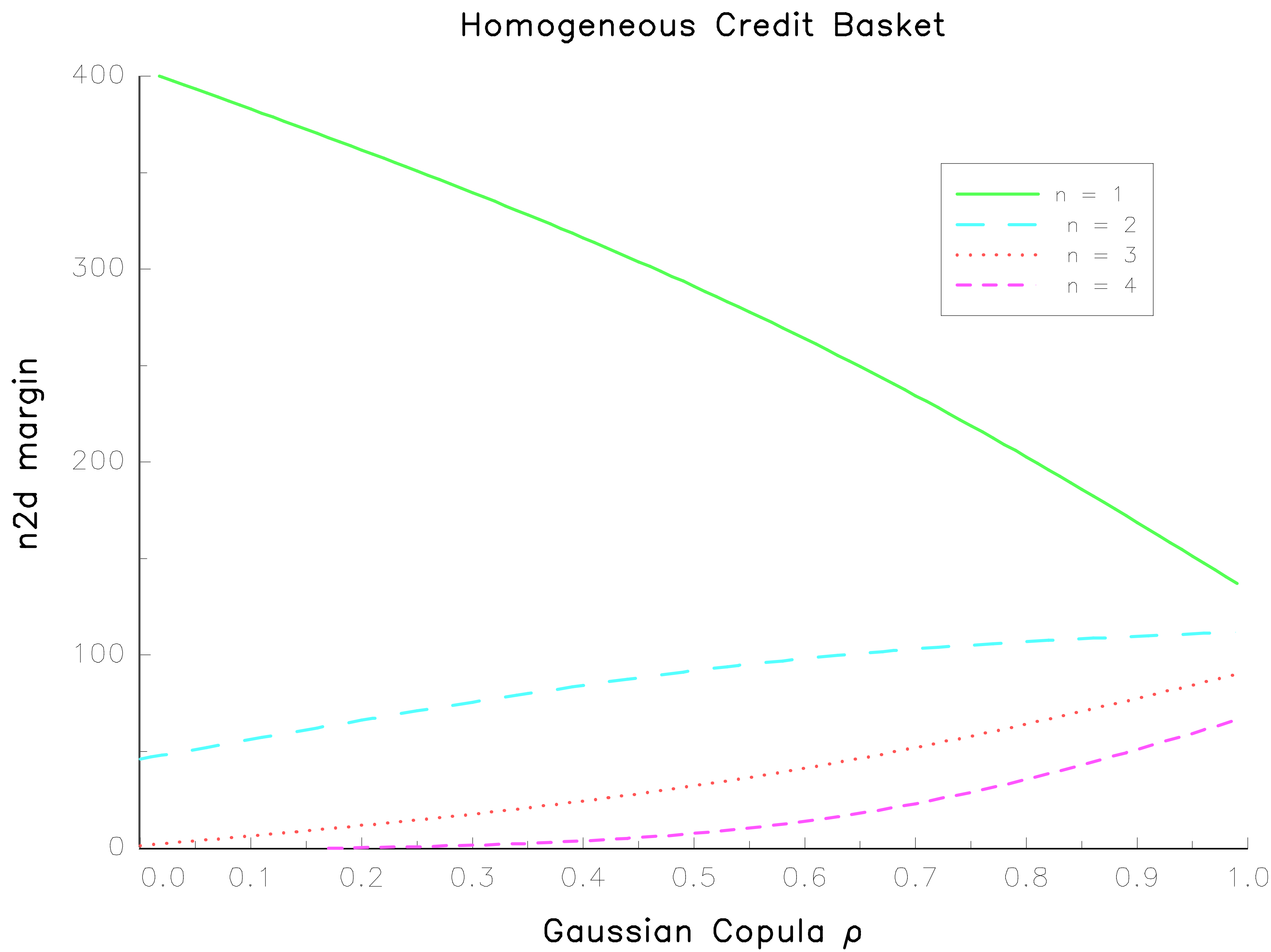
$$m_1 \approx \sum_{i=1}^{4} s_i \qquad m_2 \approx m_3 \approx m_4 \approx 0.$$

In case of the upper Fréchet copula, $\breve{\mathbf{C}}^\tau = \breve{\mathbf{C}}^+$, we have:

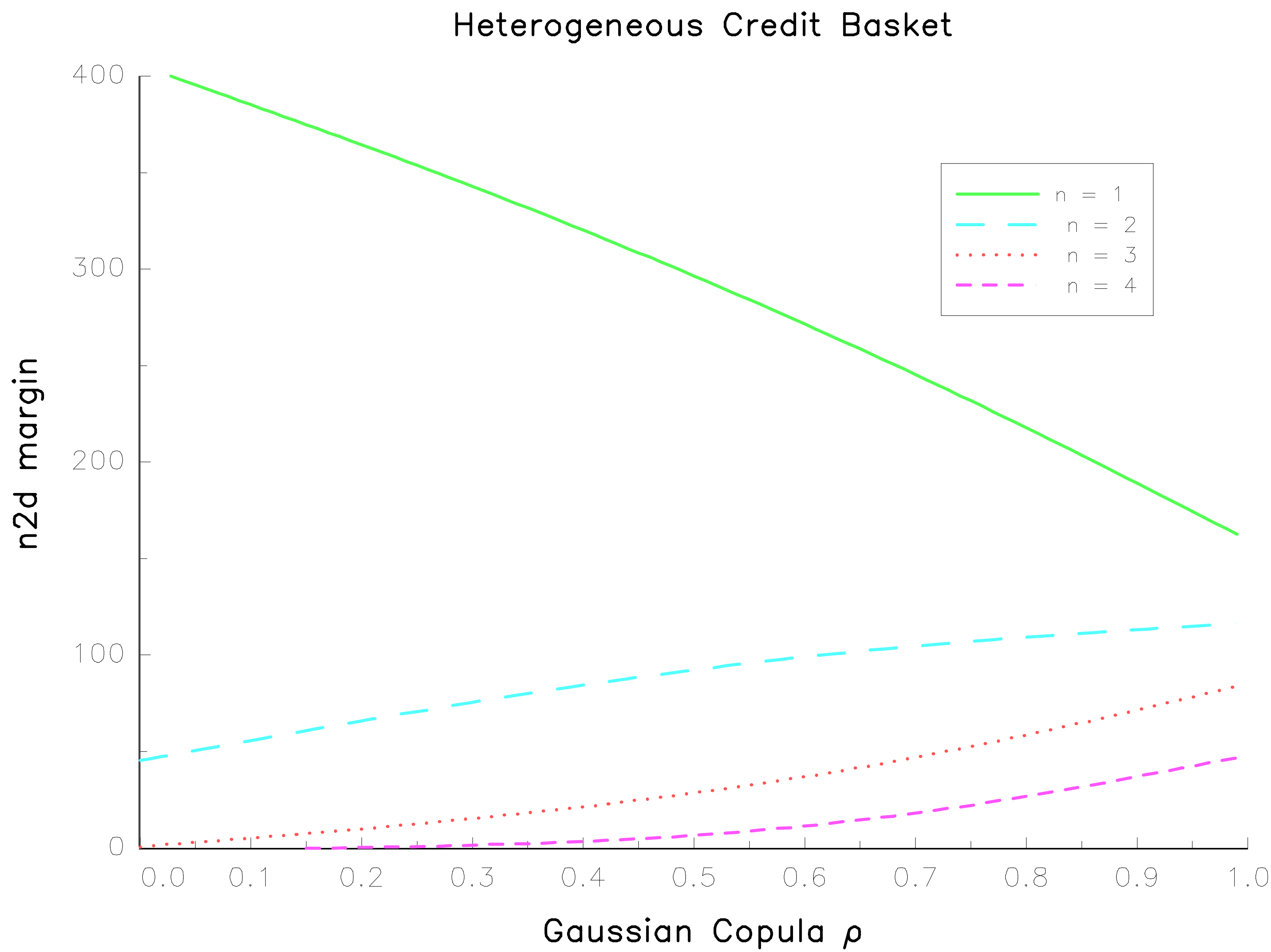
$$m_1 \approx s_{\sigma(1)}, \ldots, m_4 \approx s_{\sigma(4)}.$$

where we have sorted the corresponding intensities

$$\frac{s_{\sigma(1)}}{1 - R_{\sigma(1)}} \geq \ldots \geq \frac{s_{\sigma(4)}}{1 - R_{\sigma(4)}}$$



Homogeneous Credit Basket
n = 1
n = 2
n = 3
n = 4
400
300
200
100
0
n2d margin
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Gaussian Copula ρ


Heterogeneous Credit Basket
400
300
200
100
0
n2d margin
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Gaussian Copula ρ
n = 1
n = 2
n = 3
n = 4

# 5 Discussion Of The Model

We conclude with a survey of difficulties encountered with a daily use of the copula model.

- Why the model cannot be calibrated.
- Then how to choose the copula ?
- Other open questions.

## 5.1 Why The Model Cannot Be Calibrated

Description of a theoretical calibration procedure and why it cannot be carried out:

- Calibrating each firm's individual spread curve with Credit Default Swaps Prices.
- Choosing a copula family consistent with sectors and ratings.
- Calibrating the parameter of the copula (e.g. the correlation in case of Gaussian dependence) with the prices of First-to-default.

First-to-default market is too much illiquid to perform such a true calibration.

Another wrong ideas:

- Estimating the correlation between spreads.
- Estimating the correlation between default times.
- Estimating the jumps of spreads in case of default.
- Using Moody's Diversity Score.

## 5.2 How To Choose The Copula?

Since we do not know where the dependence is, how to choose a copula ?

- Which copula family is the right one? For the purpose of pricing we choose a copula that can easily be simulated by a Monte Carlo methodology (like Gaussian or Student).
- Which parameter is the right one? The price of the Nth-to-default contracts/CDOs is non monotonic with respect to the correlations in input, so we choose the parameter that gives the most conservative price.

We have to care for possible huge mis-pricing of credit derivatives with this method, since model risk may be very important.

## 5.3 Other Open Questions

- Repricing problem: The model is time inhomogeneous. Tomorrow the dependence between default times will be completely different from that input today.
- Hedging problem : how to hedge default risk within the copula model ?