A Tale of Two Indices

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A Tale of Two Indices: An Overview

• In 1993, the CBOE introduced the volatility index (VIX)
  – Widely followed benchmark for stock market volatility.

• On September 22, 2003, the CBOE revamped its definition of the VIX and back calculated the new VIX up to 1990.
  – The ticker for the old VIX is switched to VXO.

• The CBOE plans to launch a new Exchange, the CBOE Futures Exchange (CFE) on March 26, 2004 to trade futures and options on the new VIX.

• Questions addressed in this talk:
  – What are the differences between the two indices?
  – Why switch? Why not issue futures and options on the VXO?
  – How do VIX and VXO behave historically?
  – How do we price futures and options on VIX?
• **VXO**: Average over 8 near-the-money Black-Scholes implied volatilities at the two nearest maturities on S&P 100 index.
  
  – Essentially an estimate of the one-month at-the-money implied volatility.

• **VIX**: An average of out-of-money option *prices* \((Q)\) across all available *strikes* on S&P 500 index:

\[
\sigma^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i, T) - \frac{1}{T} \left[ \frac{F}{K_0} - 1 \right]^2, \tag{1}
\]

– Linearly interpolate (extrapolate) over the nearest two maturities to obtain a 30-day estimate, in volatility percentages.

\[
VIX = 100 \sqrt{\frac{365}{30} \left[ T_1 \sigma_1^2 \frac{N_{T_2} - 30}{N_{T_2} - N_{T_1}} + T_2 \sigma_2^2 \frac{30 - N_{T_1}}{N_{T_2} - N_{T_1}} \right]}, \tag{2}
\]
Theoretical Underpinnings of the Old VXO

- Black-Scholes implied volatility.
- Under more general settings, an accurate approximation of the volatility swap rate,
  \[ VolS \equiv \mathbb{E}_0^Q \sqrt{RV} = ATMV + O(T^3). \]  
  \[(3)\]
- The payoff of a volatility swap contract:
  \[ VolS_T = \mathbb{E}_t^Q [\sqrt{RV}_T]. \]
- The difference is converted into dollar amount based on a notional figure.
- The contract has zero market value at inception, so that
Theoretical Underpinnings of the New VIX

- Under very general settings, an approximation of the variance swap rate,
  \[ VarS \equiv \mathbb{E}_0^Q RV_T = \frac{2}{T} \int_0^\infty \frac{Q_0(K, T)}{K^2} e^{rT} dK + \varepsilon, \]  
  (4)

- The approximation error \( \varepsilon \) is zero when the futures dynamics is continuous; otherwise the error is third order \( (dF/F)^3 \).

- The payoff of a variance swap contract:
  
  \[
  \text{Variance Swap Rate (VarS)} \implies \text{Realized Variance (RV)}
  \]

- The contract has zero market value at inception, so that
  \[ VarS_T = \mathbb{E}_t^Q [RV_T]. \]
Numerical Errors Are Very Small

<table>
<thead>
<tr>
<th>ln(ν_t/θ)</th>
<th>Variance Swap</th>
<th>Volatility Swap</th>
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<tbody>
<tr>
<td></td>
<td>E_Q[RV]</td>
<td>SW</td>
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<tr>
<td>0.0</td>
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<tr>
<th>A. Black-Scholes Model</th>
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<thead>
<tr>
<th>B. Merton Jump-Diffusion (MJD) Model</th>
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<th>C. MJD-Stochastic Volatility Model</th>
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Why Switch from VXO to VIX?

- The general meaning of VXO is not known until very recently.

- The volatility swap contract underlying VXO is much more difficult to hedge than the variance swap contract underlying VIX.

- To hedge a variance swap contract, we need to take
  - A static position in a portfolio of options
  - A dynamic position in futures trading

- $VIX^2$ can also simply be regarded as the value of a portfolio of options.
VIX and VXO follow each other closely; level difference reflects mainly difference in the underlying.
### Summary Statistics

<table>
<thead>
<tr>
<th></th>
<th>VIX</th>
<th>SPX Vol</th>
<th>VXO</th>
<th>OEX Vol</th>
<th>VIX</th>
<th>SPX Vol</th>
<th>VXO</th>
<th>OEX Vol</th>
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<tr>
<td><strong>Levels</strong></td>
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<td></td>
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<tr>
<td>Mean</td>
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<td><strong>Log Levels</strong></td>
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<td>Stdev</td>
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<td>0.055</td>
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<tr>
<td>Skew</td>
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<td>0.060</td>
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<tr>
<td>Kurt</td>
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<td>-0.383</td>
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<td>-0.424</td>
<td>6.729</td>
<td>17.840</td>
<td>7.157</td>
<td>18.549</td>
</tr>
</tbody>
</table>
Cross-Correlation between Return and VIX (VXO)

Instantaneous correlation only; no lead-lag effects.
Friday average VIX is lower than averages on other weekdays. VXO does not have this effect.
The FOMC Meeting Day Effect

Vol index levels 10 days before and after FOMC meeting days.

Volatility drops by about half a percentage point after the meeting.
### The Information Content

<table>
<thead>
<tr>
<th>Series</th>
<th>Intercept</th>
<th>VIX/VXO</th>
<th>GARCH</th>
<th>R-square</th>
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<tbody>
<tr>
<td>ln VIX</td>
<td>-0.685 (0.197)</td>
<td>1.120 (0.066)</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>GARCH</td>
<td>0.180 (0.166)</td>
<td>—</td>
<td>—</td>
<td>0.907 (0.061)</td>
</tr>
<tr>
<td>Joint</td>
<td>-0.580 (0.201)</td>
<td>0.846 (0.147)</td>
<td>0.262 (0.125)</td>
<td>0.641</td>
</tr>
</tbody>
</table>

#### A. Forecasting ln $\sqrt{RV_{SPX}}$

- ln VIX
  - Intercept: -0.685 (0.197)
  - VIX/VXO: 1.120 (0.066)
  - GARCH: 0.180 (0.166)
  - R-square: 0.631

#### B. Forecasting ln $\sqrt{RV_{OEX}}$

- ln VXO
  - Intercept: -0.519 (0.177)
  - VIX/VXO: 1.065 (0.059)
  - GARCH: 0.173 (0.169)
  - R-square: 0.663

The vol indices are efficient forecasts of realized volatility; GARCH vol is not needed once the index is included.
The Excess Returns of Longing Variance Swaps

\[ \frac{(RV - VIX^2)}{VIX^2} \]

\[ \ln \frac{RV}{VIX^2} \]

Highly negative on average, but positively skewed.
The Excess Returns of Longing Volatility Swaps

\[ \frac{\sqrt{RV} - V_{XO}}{V_{XO}} \quad \frac{\ln RV}{V_{XO}} \]

Highly negative on average, but positively skewed.
VIX Futures

- The CBOE plans to launch on the VIX level (not $VIX^2$)

$$F_{0}^{vix} = E_{0}^{Q}VIX_{T_{1}},$$

(5)

- Direct valuation is not straightforward, but we obtain some bounds.

- Simplify VIX definition as a continuum of OTM options ($Q$) at one maturity

$$VIX_{T_{1}} = \sqrt{\frac{2}{T_{2} - T_{1}}}e^{r(T_{2} - T_{1})} \int_{0}^{\infty} \frac{Q_{T_{1}}(K, T_{2})}{K^{2}} dK.$$  

(6)
VIX Futures bounds

- Assume continuous futures price dynamics, we have

$$VIX_{T_1} = \sqrt{E^Q_{T_1} RV_{T_1,T_2}},$$

where $RV_{T_1,T_2}$ is the annualized return quadratic variation between $T_1$ and $T_2$.

- The VIX futures can be represented as:

$$F_{0}^{vix} = E^Q_0 \sqrt{E^Q_{T_1} RV_{T_1,T_2}},$$

- The concavity of the square root and Jensen’s inequality implies that:

$$E^Q_0 \sqrt{RV_{T_1,T_2}} \leq F_{0}^{vix} \leq \sqrt{E^Q_0 RV_{T_1,T_2}},$$

by the law of iterated expectations.
The Upper Bound for the VIX Futures

\[ E_0^Q \sqrt{RV_{T_1,T_2}} \leq F_{0}^{vix} \leq \sqrt{E_0^Q RV_{T_1,T_2}}. \]

- The upper bound is the square root of the forward variance swap rate,

\[ U_0 \equiv \sqrt{E_0^Q RV_{T_1,T_2}}, \quad (10) \]

- Can be represented as a function of European option prices on SPX

\[ U_0^2 = E_0^Q RV_{T_1,T_2} = E_0^Q RV_{0,T_2} - E_0^Q RV_{0,T_1} \]

\[ = \frac{1}{T_2 - T_1} \int_0^\infty \left[ Q_0(K,T_2)e^{rT_2} - Q_0(K,T_1)e^{rT_1} \right] \frac{2}{K^2} dK. \quad (11) \]

- Hence observable.
The Lower Bound for the VIX Futures

\[ E_0^Q \sqrt{RV_{T_1,T_2}} \leq F^\text{vix}_0 \leq \sqrt{E_0^Q RV_{T_1,T_2}}. \]

• The lower bound is essentially the forward vol swap rate.

\[ L_0 \equiv E_0^Q \sqrt{RV_{T_1,T_2}}. \]  \hfill (12)

• Under more assumptions, it can be approximated by a forward-start at-the-money forward call option price (FSATMFC),

\[ L_0 \approx \sqrt{2\pi} \frac{FSATMFC_0(T_1, T_2)e^{rT_2}}{\sqrt{T_2 - T_1}}, \]  \hfill (13)

• These forward-start options are active over the counter, hence observable again.
How Tight are the Bounds?

Moves with the vol level; between 0.5-3.5 percentage vol points.
• Recall that no-arbitrage implies

\[ E_0^Q \sqrt{RV_{T_1,T_2}} \leq F^{vix}_0 \leq \sqrt{E_0^Q RV_{T_1,T_2}} \equiv U_0. \]

• Suppose \( F^{vix} \geq U_0 \). Do this
  - Buy low implies: Buy one unit of forward variance swap.
  - Sell High implies: Sell \( 2F^{vix}_0 \) units of VIX futures.

• At the VIX futures maturity \( T_1 \), VIX futures \( (F^{vix}_{T_1}) \) converge to \( VIX_{T_1} \). Forward variance swap \( (U^2_{T_1}) \) converges to \( VIX^2_{T_1} \).
P&L from the Arb

• Recall that $F^{vix} \geq U_0$.

• Recall our position: Long one forward variance swap; short $2F_0^{vix}$ VIX futures.

• Our P&L at VIX futures maturity $T_1$ is

$$PL = \text{VIX}^2_{T_1} - U_0^2 - 2F_0^{vix}[\text{VIX}_{T_1} - F_0^{vix}]$$

$$= \text{VIX}^2_{T_1} - 2F_0^{vix}\text{VIX}_{T_1} + 2(F_0^{vix})^2 - U_0^2$$

$$\geq \text{VIX}^2_{T_1} - 2F_0^{vix}\text{VIX}_{T_1} + (F_0^{vix})^2$$

$$= (\text{VIX}_{T_1} - F_0^{vix})^2 \geq 0.$$
• Recall that no-arbitrage implies

\[ L_0 \equiv E_Q^0 \sqrt{RV_{T_1,T_2}} \leq F^{vix}_0 \leq \sqrt{E_Q^0 RV_{T_1,T_2}}. \]

• Suppose \( F^{vix}_0 \leq L_0 \). Do this
  – Buy low implies: Buy one unit of VIX futures.
  – Sell High implies: Sell one unit of forward vol swap.

• At the VIX futures maturity \( T_1 \), VIX futures \( (F^{vix}_{T_1}) \) converge to \( VIX_{T_1} \).
  Forward vol swap \( (L_{T_1}) \) converges to a spot vol swap rate \( (VolS_{T_1}) \).

• Recall that vol swap rate is always lower than the variance swap rate:
  \( VolS_{T_1} \leq VIX_{T_1} \).
P&L from the Arb

• Recall that $F^{vix} \leq L_0$.

• Recall our position: Long one VIX futures; short one forward vol swap.

• Recall also: $VolS \leq VIX$.

• Our P&L at VIX futures maturity $T_1$ is

\[
PL = VIX_{T_1} - F^{vix}_0 - [VolS_{T_1} - L_0] \\
= [VIX_{T_1} - VolS_{T_1}] + [L_0 - F^{vix}_0] \geq 0.
\]
The Economic Meanings of the Bounds

Apply the following equality, \( \text{var}(x) = E[x^2] - (E[x])^2 \)

- The risk-neutral variance of the future volatility

\[
\text{Var}_Q^0 \sqrt{RV_{T_1,T_2}} = E_Q^0 RV_{T_1,T_2} - \left( E_Q^0 \sqrt{RV_{T_1,T_2}} \right)^2 = U_0^2 - L_0^2. \tag{14}
\]

- The risk-neutral variance of the future expected volatility (VIX)

\[
\text{Var}_Q^0 \sqrt{E_{T_1}^Q RV_{T_1,T_2}} = E_Q^0 RV_{T_1,T_2} - \left( E_Q^0 \sqrt{E_{T_1}^Q RV_{T_1,T_2}} \right)^2 = U_0^2 - (F_{vix}^0)^2. \tag{15}
\]
Pricing VIX Options

• The terminal payoff of a call option on the VIX is \((VIX_{T_1} - K)^+\)

• We hence need to know the risk-neutral distribution of VIX for valuation.

• We do not know the distribution, but we (will) observe the first two risk-neutral moments of \(VIX_{T_1}\),

  \[
  \text{Mean} : \quad E_Q^0 [VIX_{T_1}] = F_0^{\text{vix}} = F_0 \\
  \text{Variance} : \quad Var_Q^0 [VIX_{T_1}] = U_0^2 - (F_0^{\text{vix}})^2 \equiv V_0
  \]

• We can price options with a distributional assumption.
Pricing VIX Options under Black-Scholes

• The Black-Scholes model

\[ C_0 = e^{-rT_1} \left[ F_0 N(d_1) - KN(d_2) \right], \]

where

\[ d_1 = \frac{\ln F_0 / K + \frac{1}{2} s^2 T_1}{s \sqrt{T_1}}, \quad d_2 = d_1 - s \sqrt{T_1}, \]

and \( s \) is the annualized standard deviation of \( \ln VIX_{T_1} \), which can be represented as a function of the first two moments of \( VIX_{T_1} \),

\[ s = \sqrt{\frac{1}{T_1} \ln \frac{V_0 + F_0^2}{F_0^2}}. \] (16)

• The key is the observable input for \( V_0 \), and hence for \( s \).
Pricing VIX Options under Bachelier

- Under a normal assumption, the call option value is
  \[ C_0 = e^{-rT_1} \left[ \sqrt{V_0} N'(d) + (F_0 - K) N(d) \right], \]
  with
  \[ d = \frac{F_0 - K}{\sqrt{V_0}}. \]

- The key is the observable input for \( F_0 \) and for \( V_0 \).
The VIX approximates the variance swap rate on SPX; the old VXO approximates the volatility swap rate on OEX.

Payoffs to variance swap contracts are much easier to replicate/hedge than payoffs to vol swaps. Hence, CBOE switched to the new VIX and plans to launch futures and options on the new VIX.

We provide tight bounds on the VIX futures.

We can also price VIX options using observables.

Looking forward to a fascinating market...