Coupled Diffusions and Systemic Risk

"Systemic Risk Illustrated"

Jean-Pierre Fouque

University of California Santa Barbara

Joint work with Li-Hsien Sun

Meeting on Financial Risks

RiskLab

Madrid, May 24th, 2012

HANDBOOK ON SYSTEMIC RISK

Editors: J.-P. Fouque and J. Langsam

Cambridge University Press (to appear in 2012)

HANDBOOK ON SYSTEMIC RISK

Editors: J.-P. Fouque and J. Langsam
Cambridge University Press (to appear in 2012)

- 2008: Committee to establish a **National Institute of Finance** (data and research on systemic risk)

HANDBOOK ON SYSTEMIC RISK

Editors: J.-P. Fouque and J. Langsam
Cambridge University Press (to appear in 2012)

- 2008: Committee to establish a **National Institute of Finance** (data and research on systemic risk)
- 2010: Dodd-Frank Bill includes the creation of the **Office of Financial Research**

HANDBOOK ON SYSTEMIC RISK

Editors: J.-P. Fouque and J. Langsam
Cambridge University Press (to appear in 2012)

- 2008: Committee to establish a **National Institute of Finance** (data and research on systemic risk)
- 2010: Dodd-Frank Bill includes the creation of the **Office of Financial Research**
- 2012: Director nominated: **Richard Berner** will form a **Financial Research Advisory Committee**

Correlated Diffusions: Credit Risk

$Y_t^{(i)}, i = 1, \ldots, N$ denote log-values

$$dY_t^{(i)} = b_t^{(i)} dt + \sigma_t^{(i)} dW_t^{(i)} \qquad i = 1, \ldots, N.$$

Three ingredients:

- Drifts $b_t^{(i)}$
- Volatilities $\sigma_t^{(i)}$
- Brownian motions $W_t^{(i)}$

Correlated Diffusions: Credit Risk

$Y_t^{(i)}, i = 1, \ldots, N$ denote log-values

$$dY_t^{(i)} = b_t^{(i)} dt + \sigma_t^{(i)} dW_t^{(i)} \qquad i = 1, \ldots, N.$$

Three ingredients:

- Drifts $b_t^{(i)}$
- Volatilities $\sigma_t^{(i)}$
- Brownian motions $W_t^{(i)}$

Credit Risk (structural approach):

drifts imposed by risk neutrality

Correlated Diffusions: Credit Risk

$Y_t^{(i)}, i = 1, \ldots, N$ denote log-values

$$dY_t^{(i)} = b_t^{(i)} dt + \sigma_t^{(i)} dW_t^{(i)} \qquad i = 1, \ldots, N.$$

Three ingredients:

- Drifts $b_t^{(i)}$
- Volatilities $\sigma_t^{(i)}$
- Brownian motions $W_t^{(i)}$

Credit Risk (structural approach):

drifts imposed by risk neutrality

Correlation is created between the BMs

Joint distribution of hitting times is a problem!

Correlated Diffusions: Credit Risk

$Y_t^{(i)}, i = 1, \ldots, N$ denote log-values

$$dY_t^{(i)} = b_t^{(i)} dt + \sigma_t^{(i)} dW_t^{(i)} \qquad i = 1, \ldots, N.$$

Three ingredients:

- Drifts $b_t^{(i)}$
- Volatilities $\sigma_t^{(i)}$
- Brownian motions $W_t^{(i)}$

Credit Risk (structural approach):

drifts imposed by risk neutrality

Correlation is created between the BMs

Joint distribution of hitting times is a problem!

Correlation can also be created through stochastic volatilities $\sigma_t^{(i)}$

(Fouque-Wignall-Zhou 2008)

Coupled Diffusions: Systemic Risk

$Y_t^{(i)}, i = 1, \ldots, N$ denote log-monetary reserves of N banks

$$dY_t^{(i)} = b_t^{(i)} dt + \sigma_t^{(i)} dW_t^{(i)} \qquad i = 1, \ldots, N.$$

Coupled Diffusions: Systemic Risk

$Y_t^{(i)}, i = 1, \ldots, N$ denote log-monetary reserves of N banks

$$dY_t^{(i)} = b_t^{(i)} dt + \sigma_t^{(i)} dW_t^{(i)} \qquad i = 1, \ldots, N.$$

Assume **independent Brownian motions** $W_t^{(i)}, i = 1, \ldots, N$ and **identical constant volatilities** $\sigma_t^{(i)} = \sigma$

Coupled Diffusions: Systemic Risk

$Y_t^{(i)}, i = 1, \ldots, N$ denote log-monetary reserves of N banks

$$dY_t^{(i)} = b_t^{(i)} dt + \sigma_t^{(i)} dW_t^{(i)} \qquad i = 1, \ldots, N.$$

Assume **independent Brownian motions** $W_t^{(i)}, i = 1, \ldots, N$
and **identical constant volatilities** $\sigma_t^{(i)} = \sigma$
Model **borrowing and lending** through the drifts:

$$dY_t^{(i)} = \frac{\alpha}{N} \sum_{j=1}^{N} (Y_t^{(j)} - Y_t^{(i)})\, dt + \sigma dW_t^{(i)}, \quad i = 1, \ldots, N.$$

The overall **rate of borrowing and lending** α/N has been normalized by the number of banks and we assume $\alpha > 0$

Coupled Diffusions: Systemic Risk

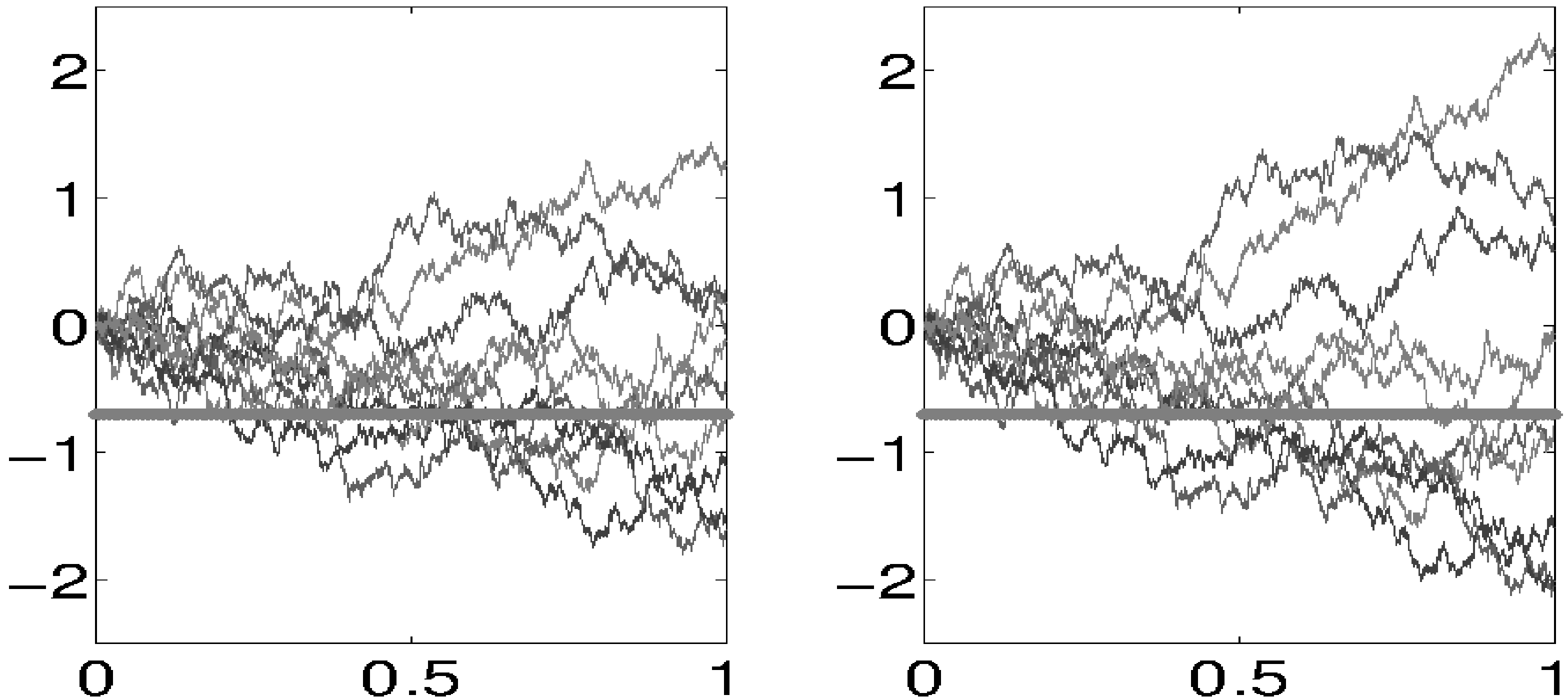
$Y_t^{(i)}, i = 1, \ldots, N$ denote log-monetary reserves of N banks

$$dY_t^{(i)} = b_t^{(i)} dt + \sigma_t^{(i)} dW_t^{(i)} \qquad i = 1, \ldots, N.$$

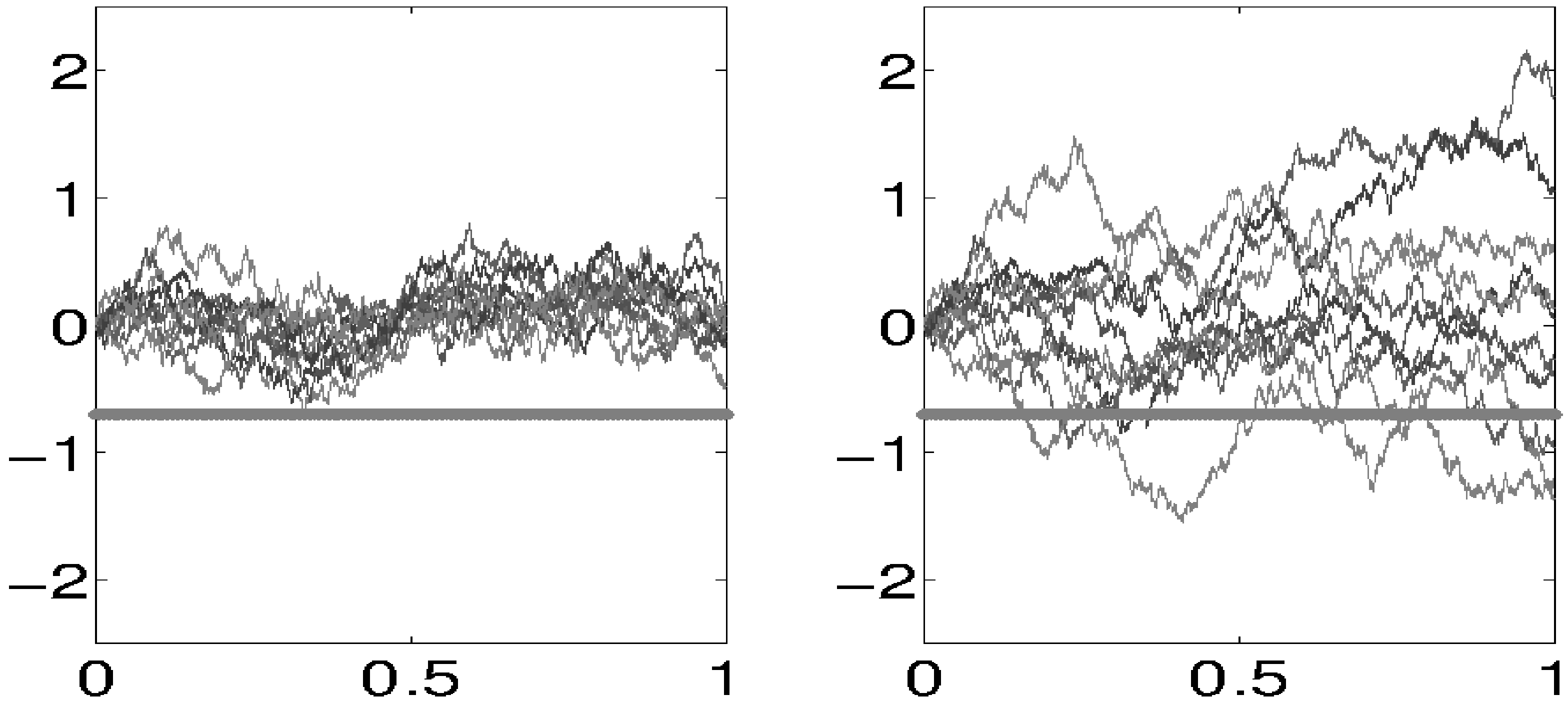
Assume **independent Brownian motions** $W_t^{(i)}, i = 1, \ldots, N$
and **identical constant volatilities** $\sigma_t^{(i)} = \sigma$
Model **borrowing and lending** through the drifts:

$$dY_t^{(i)} = \frac{\alpha}{N} \sum_{j=1}^{N} (Y_t^{(j)} - Y_t^{(i)})\, dt + \sigma dW_t^{(i)}\,, \quad i = 1, \ldots, N.$$

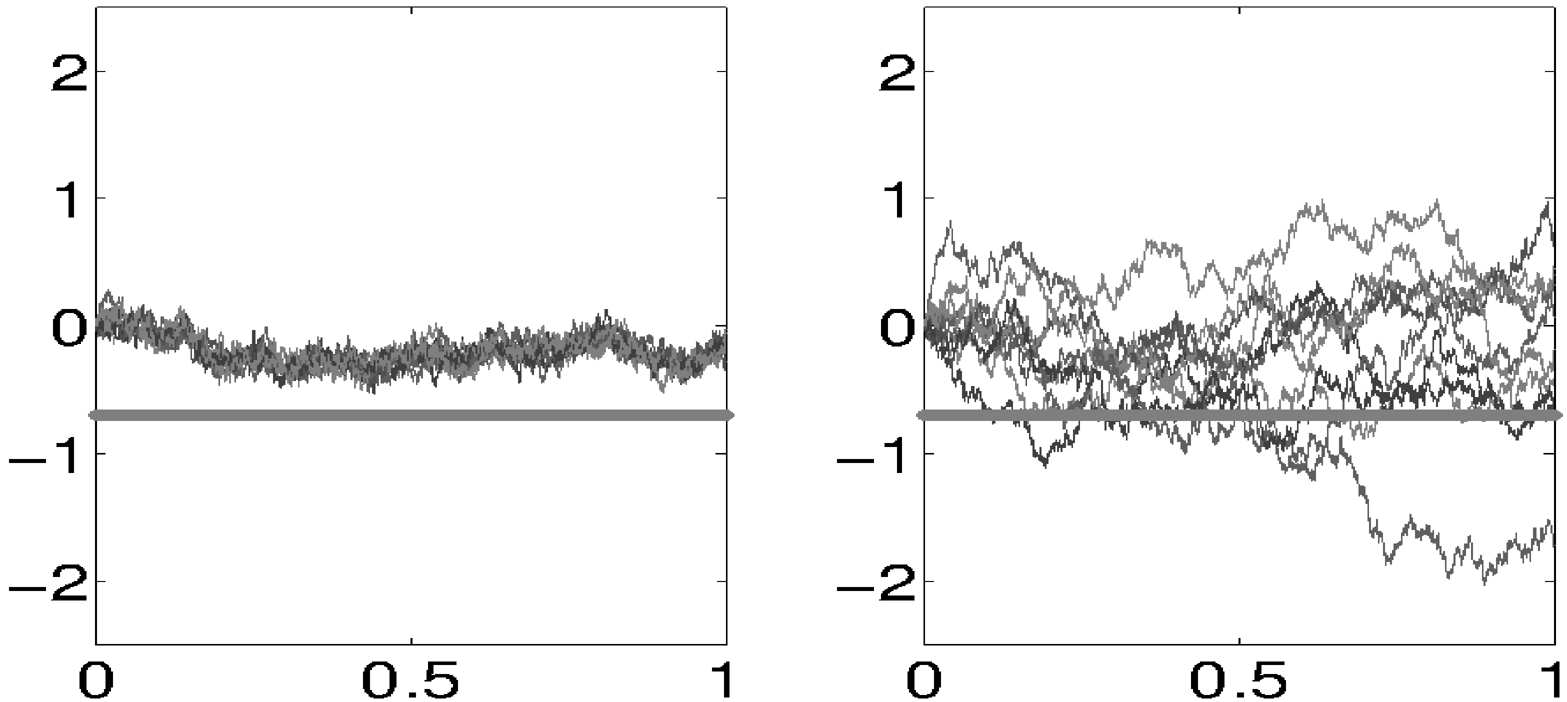
The overall **rate of borrowing and lending** α/N has been normalized by the number of banks and we assume $\alpha > 0$
Denote the **default level** by $\eta < 0$ and simulate the system for various values of α: $\mathbf{0}, \mathbf{1}, \mathbf{10}, \mathbf{100}$ with fixed $\sigma = 1$



One realization of the trajectories of the coupled diffusions $Y_t^{(i)}$ with $\alpha = \mathbf{1}$ (left plot) and trajectories of the independent Brownian motions ($\alpha = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $\eta = -0.7$



One realization of the trajectories of the coupled diffusions $Y_t^{(i)}$ with $\alpha = \mathbf{10}$ (left plot) and trajectories of the independent Brownian motions ($\alpha = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $\eta = -0.7$



One realization of the trajectories of the coupled diffusions $Y_t^{(i)}$ with $\underline{\boldsymbol{\alpha = 100}}$ (left plot) and trajectories of the independent Brownian motions ($\alpha = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $\eta = -0.7$

These simulations "show" that STABILITY is created by increasing the rate α of borrowing and lending.

These simulations “show” that STABILITY is created by increasing the rate α of borrowing and lending.

Next, we compare the **loss distributions** for the coupled and independent cases. We compute these loss distributions by Monte Carlo method using 10^4 simulations, and with the same parameters as previously.

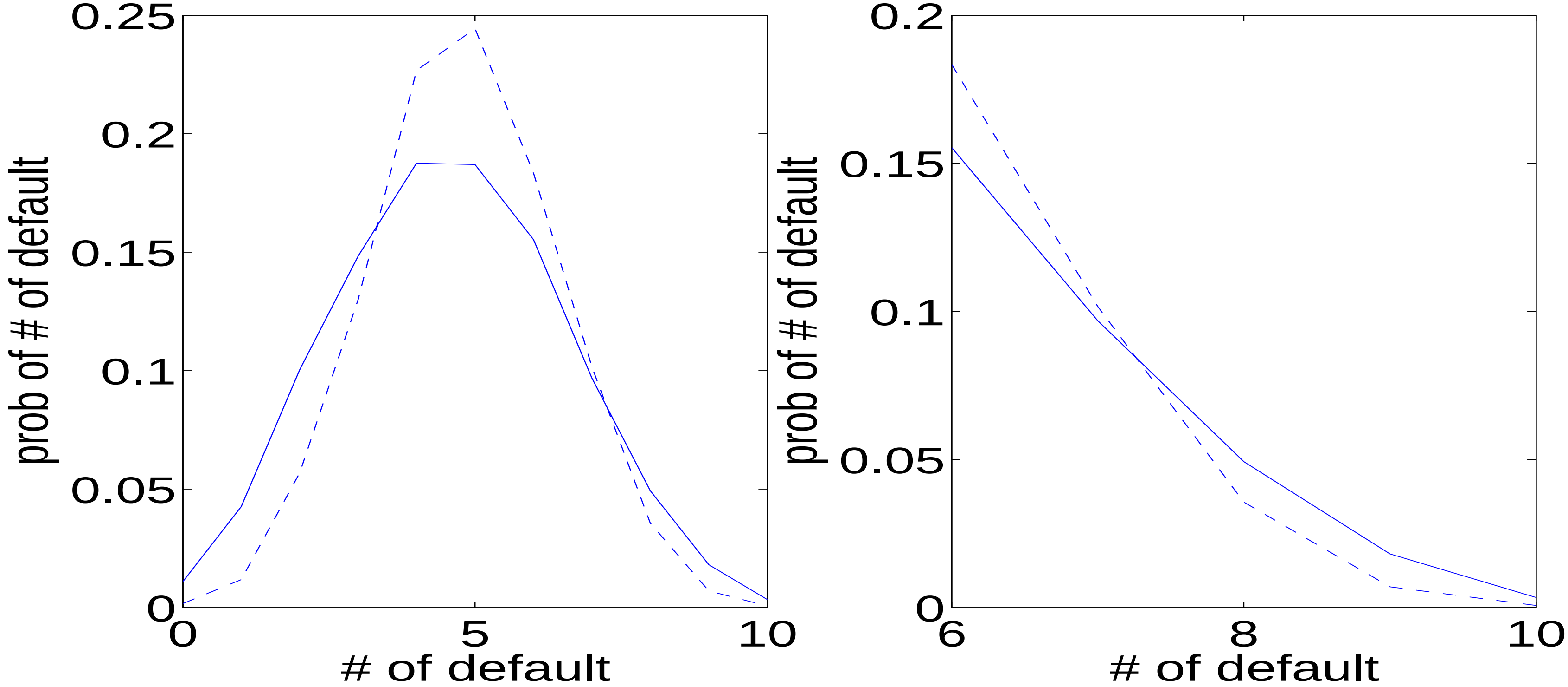
These simulations "show" that STABILITY is created by increasing the rate α of borrowing and lending.

Next, we compare the **loss distributions** for the coupled and independent cases. We compute these loss distributions by Monte Carlo method using 10^4 simulations, and with the same parameters as previously.

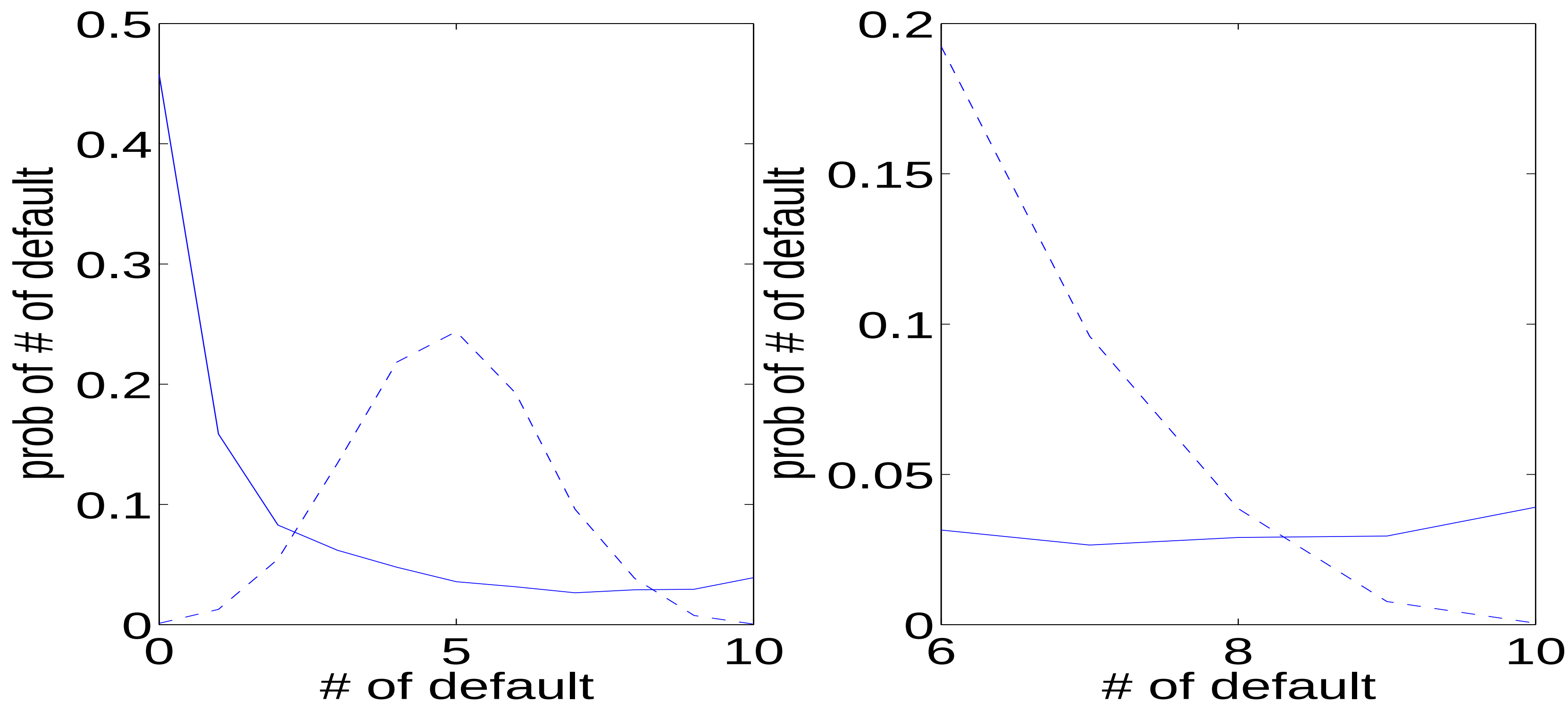
In the independent case, the loss distribution is Binomial(N, p) with parameter p given by

$$\begin{aligned} p &= \mathbb{P}\left(\min_{0 \leq t \leq T} (\sigma W_t) \leq \eta \right) \\ &= 2\Phi\left(\frac{\eta}{\sigma\sqrt{T}} \right), \end{aligned}$$

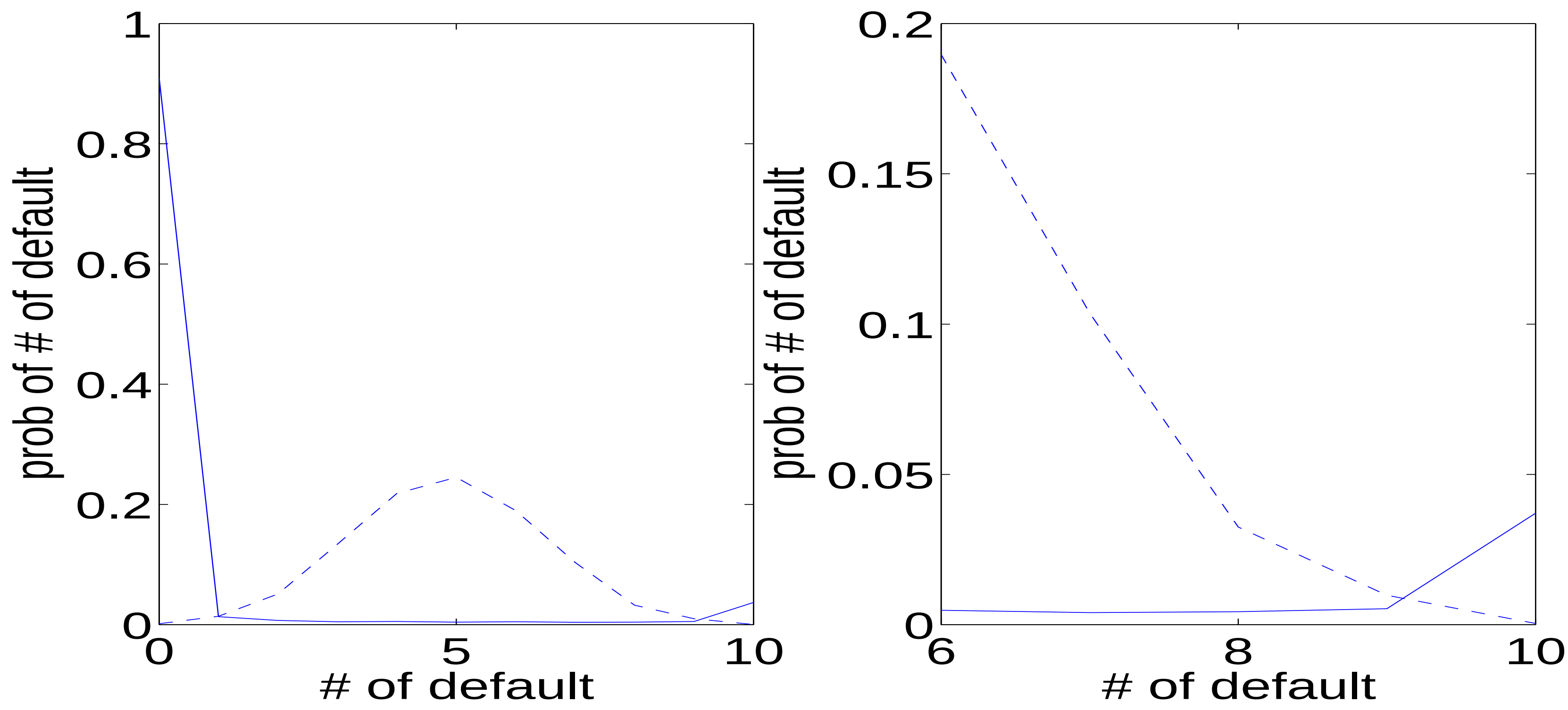
where Φ denotes the $\mathcal{N}(0,1)$-cdf, and we used the distribution of the minimum of a Brownian motion (see Karatzas-Shreve 2000 for instance). With our choice of parameters, we have $p \approx 0.5$



On the left, we show plots of the loss distribution for the coupled diffusions with $\alpha = \mathbf{1}$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.



On the left, we show plots of the loss distribution for the coupled diffusions with $\alpha = \mathbf{10}$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.



On the left, we show plots of the loss distribution for the coupled diffusions with $\alpha = \mathbf{100}$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.

Mean-field Limit

Rewrite the dynamics as:

$$\begin{aligned} dY_t^{(i)} &= \frac{\alpha}{N}\sum_{j=1}^{N}(Y_t^{(j)} - Y_t^{(i)})\,dt + \sigma dW_t^{(i)} \\ &= \alpha\left[\left(\frac{1}{N}\sum_{j=1}^{N} Y_t^{(j)}\right) - Y_t^{(i)}\right]dt + \sigma dW_t^{(i)}. \end{aligned}$$

Mean-field Limit

Rewrite the dynamics as:

$$
\begin{aligned}
dY_t^{(i)} &= \frac{\alpha}{N}\sum_{j=1}^{N}(Y_t^{(j)} - Y_t^{(i)})\,dt + \sigma dW_t^{(i)} \\
&= \alpha\left[\left(\frac{1}{N}\sum_{j=1}^{N} Y_t^{(j)}\right) - Y_t^{(i)}\right] dt + \sigma dW_t^{(i)}.
\end{aligned}
$$

The processes $Y^{(i)}$'s are "OUs" **mean-reverting** to the **ensemble average** which satisfies

$$
d\left(\frac{1}{N}\sum_{i=1}^{N} Y_t^{(i)}\right) = d\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}\right).
$$

Assuming for instance that $y_0^{(i)} = 0$, $i = 1, \ldots, N$, we obtain

$$\frac{1}{N}\sum_{i=1}^{N} Y_t^{(i)} = \frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}, \quad \text{and consequently}$$

$$dY_t^{(i)} = \alpha\left[\left(\frac{\sigma}{N}\sum_{j=1}^{N} W_t^{(j)}\right) - Y_t^{(i)}\right] dt + \sigma dW_t^{(i)}\,.$$

Assuming for instance that $y_0^{(i)} = 0$, $i = 1, \ldots, N$, we obtain

$$\frac{1}{N}\sum_{i=1}^{N} Y_t^{(i)} = \frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}, \quad \text{and consequently}$$

$$dY_t^{(i)} = \alpha\left[\left(\frac{\sigma}{N}\sum_{j=1}^{N} W_t^{(j)}\right) - Y_t^{(i)}\right] dt + \sigma dW_t^{(i)}\,.$$

Note that the ensemble average is distributed as a Brownian motion with diffusion coefficient $\sigma/\sqrt{N}$.

Assuming for instance that $y_0^{(i)} = 0$, $i = 1, \ldots, N$, we obtain

$$\frac{1}{N}\sum_{i=1}^{N} Y_t^{(i)} = \frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}, \quad \text{and consequently}$$

$$dY_t^{(i)} = \alpha \left[\left(\frac{\sigma}{N}\sum_{j=1}^{N} W_t^{(j)} \right) - Y_t^{(i)} \right] dt + \sigma dW_t^{(i)} .$$

Note that the ensemble average is distributed as a Brownian motion with diffusion coefficient $\sigma/\sqrt{N}$.

In the limit $N \to \infty$, the strong law of large numbers gives

$$\frac{1}{N}\sum_{j=1}^{N} W_t^{(j)} \to 0 \quad a.s. ,$$

and therefore, the processes $Y^{(i)}$'s converge to independent OU processes with long-run mean zero.

In fact, $Y_t^{(i)}$ is given explicitly by

$$Y_t^{(i)} = \frac{\sigma}{N}\sum_{j=1}^{N} W_t^{(j)} + \sigma e^{-\alpha t}\int_0^t e^{\alpha s} dW_s^{(i)} - \frac{\sigma}{N}\sum_{j=1}^{N}\left(e^{-\alpha t}\int_0^t e^{\alpha s} dW_s^{(j)}\right),$$

and therefore, $Y_t^{(i)}$ converges to $\sigma e^{-\alpha t}\int_0^t e^{\alpha s} dW_s^{(i)}$ which are independent OU processes.

This is a simple example of a **mean-field limit** and propagation of chaos studied in general by Sznitman (1991).

Large Deviation

We focus on the event where the ensemble average reaches the default level. The probability of this event is small (as N becomes large), and is given by the theory of **Large Deviation**.

Large Deviation

We focus on the event where the ensemble average reaches the default level. The probability of this event is small (as N becomes large), and is given by the theory of **Large Deviation**.

In our simple example, this probability can be computed explicitly as follows:

$$\begin{aligned} I\!P\left(\min_{0\le t\le T}\left(\frac{\sigma}{N}\sum_{i=1}^{N}W_t^{(i)}\right)\le\eta\right) &= I\!P\left(\min_{0\le t\le T}\widetilde{W}_t\le\frac{\eta\sqrt{N}}{\sigma}\right) \\ &= 2\Phi\left(\frac{\eta\sqrt{N}}{\sigma\sqrt{T}}\right), \end{aligned}$$

where $\widetilde{W}$ is a standard Brownian motion.

Systemic Risk

Using classical equivalent for the Gaussian cumulative distribution function, we obtain

$$\lim_{N\to\infty} -\frac{1}{N}\log \mathbb{P}\left(\min_{0\le t\le T}\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}\right) \le \eta\right) = \frac{\eta^2}{2\sigma^2 T}\,.$$

Systemic Risk

Using classical equivalent for the Gaussian cumulative distribution function, we obtain

$$\lim_{N\to\infty} -\frac{1}{N}\log \mathbb{P}\left(\min_{0\le t\le T}\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}\right)\le \eta\right) = \frac{\eta^2}{2\sigma^2 T}\,.$$

For a large number of banks, the probability that the ensemble average reaches the default barrier is of order $\exp\left(-\frac{\eta^2 N}{2\sigma^2 T}\right)$

Systemic Risk

Using classical equivalent for the Gaussian cumulative distribution function, we obtain

$$\lim_{N\to\infty} -\frac{1}{N}\log I\!\!P\left(\min_{0\le t\le T}\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}\right)\le\eta\right)=\frac{\eta^2}{2\sigma^2 T}\,.$$

For a large number of banks, the probability that the ensemble average reaches the default barrier is of order $\exp\left(-\frac{\eta^2 N}{2\sigma^2 T}\right)$

$$\text{Since}\quad \frac{1}{N}\sum_{i=1}^{N} Y_t^{(i)}=\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)},\quad \text{we identify}$$

$$\left\{\min_{\mathbf{0\le t\le T}}\left(\frac{\sigma}{\mathbf{N}}\sum_{\mathbf{i=1}}^{\mathbf{N}}\mathbf{Y_t^{(i)}}\right)\le\eta\right\}\quad \text{as a } \textbf{systemic event}$$

Observe that this event does not depend on $\alpha > 0$

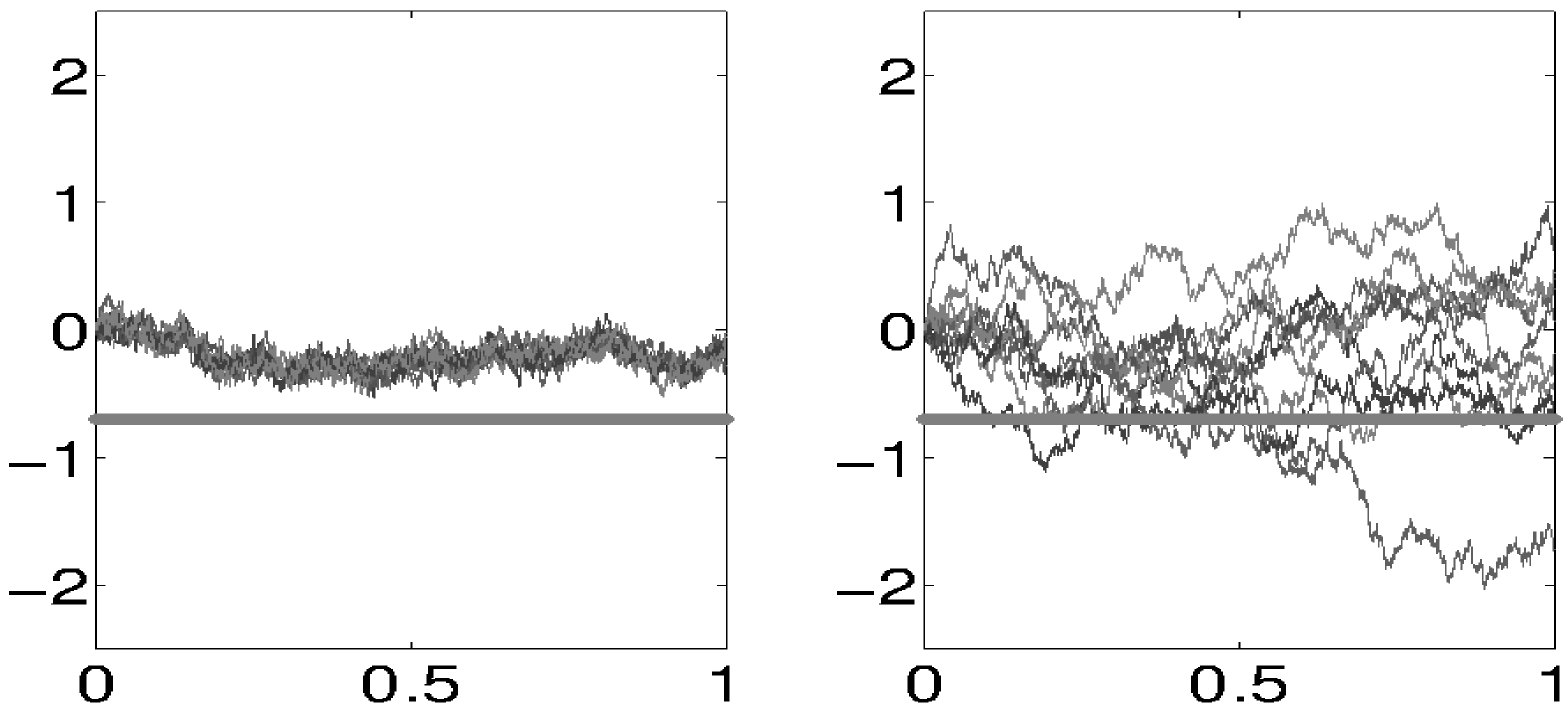
The probability

$$\exp\left(-\frac{\eta^2 N}{2\sigma^2 T}\right)$$

of a systemic event does not depend on $\alpha > 0$, in other words:

" Increasing stability by increasing the rate of borrowing and lending does not prevent a systemic event where a large number of banks default"

In fact, once in this event, increasing α creates even more defaults by **"flocking to default"**. This is illustrated in the simulation with $\alpha = 100$ where the probability of systemic risk is roughly 3%.



One realization of the trajectories of the coupled diffusions $Y_t^{(i)}$ with $\underline{\alpha = \mathbf{100}}$ (left plot) and trajectories of the independent Brownian motions ($\alpha = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $\eta = -0.7$.

The probability of a systemic event is roughly 3%

Summary

We proposed a simple toy model of coupled diffusions to represent **lending and borrowing** between banks. We show that, as expected, this activity **stabilizes** the system in the sense that it decreases the number of defaults. Indeed, and naively, banks in difficulty can be "saved" by borrowing from others. In fact, the model illustrates the fact that stability increases as the rate of borrowing and lending increases.

However, there is a small probability, computed explicitly in our model, that the average of the ensemble reaches the default level. Combined with the "flocking" behavior **"everybody follows everybody"**, this leads to a **systemic event** where almost all default, in particular when the rate of borrowing and lending is large.

Related Papers

- *Diversification in Financial Networks may Increase Systemic Risk*

  by J. Garnier, G. Papanicolaou, and T.-W. Yang

  To appear in the Handbook of Systemic Risk (2012)

- *Stability in a model of inter-bank lending*

  by J.-P. Fouque and T. Ichiba (Submitted).

Related Papers

- *Diversification in Financial Networks may Increase Systemic Risk*

  by J. Garnier, G. Papanicolaou, and T.-W. Yang

  To appear in the Handbook of Systemic Risk (2012)

- *Stability in a model of inter-bank lending*

  by J.-P. Fouque and T. Ichiba (Submitted).

On Credit Risk:

- *Modeling Correlated Defaults: First Passage Model under Stochastic Volatility* (by J.-P. Fouque, B. Wignall and X. Zhou). Journal of Computational Finance **11**(3), 2008.

- *Multiscale Stochastic Volatility for Equity, Interest-Rate and Credit Derivatives* (**BOOK** by J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Sølna). Cambridge University Press 2011.

“Lending and borrowing improves stability but also contributes to systemic risk”

THANKS FOR YOUR ATTENTION