Beyond the single-loss approximation in operational risk.

Alberto Suárez[*, Lexuri Fernández[**]

Computer Science Department, Universidad Autónoma de Madrid

[*, Quantitative Risk Research (QRR) RiskLab Madrid

[**] Universidad del País Vasco

alberto.suarez@uam.es
Extreme events in finance

Modeling of extreme events is possible and provides insight

- Rare events
- Large impact
- Difficult to predict their specific characteristics (but can be explained after the fact).
- Amenable to modeling
Beyond single-loss in Operational Risk
The three pillars approach

- **First pillar: Minimum capital requirements** (quantification of risk)
  - Specifies the guiding principles for the estimation of regulatory/economic capital.
  - Operational risk is included as a new type of risk.

- **Second pillar: Supervisory review process.**

- **Third pillar: Market discipline (+ public disclosure)**
Operational Risk: Definition

[source: Basel II]

644. Operational risk is defined as the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events. This definition includes legal risk, but excludes strategic and reputational risk.
## Operational Risk: Measurement

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>V.</td>
<td>Operational Risk</td>
<td>144</td>
</tr>
<tr>
<td>A.</td>
<td>Definition of operational risk</td>
<td>144</td>
</tr>
<tr>
<td>B.</td>
<td>The measurement methodologies</td>
<td>144</td>
</tr>
<tr>
<td></td>
<td>1. The Basic Indicator Approach</td>
<td>144</td>
</tr>
<tr>
<td></td>
<td>2. The Standardised Approach</td>
<td>146</td>
</tr>
<tr>
<td></td>
<td>3. Advanced Measurement Approaches (AMA)</td>
<td>147</td>
</tr>
<tr>
<td>C.</td>
<td>Qualifying criteria</td>
<td>148</td>
</tr>
<tr>
<td></td>
<td>1. The Standardised Approach</td>
<td>148</td>
</tr>
<tr>
<td></td>
<td>2. Advanced Measurement Approaches (AMA)</td>
<td>149</td>
</tr>
<tr>
<td>D.</td>
<td>Partial use</td>
<td>156</td>
</tr>
</tbody>
</table>
Measurement approaches

- **Basic Indicator Approach (BIA)**
  
  \[ K_{\text{BIA}} = \alpha \times EI, \quad \text{where} \quad \begin{cases} \alpha = 0.15 \\ EI = \text{gross income (mean of the last 3 years)} \end{cases} \]

- **Standardised Approach (TSA)**
  
  \[ K_{\text{TSA}} = \sum_{i=1}^{8} \beta_i \times EI_i, \quad \text{where} \quad \begin{cases} \beta_i \text{ are defined by the regulator} \\ EI_i \text{ are the gross income for line i} \end{cases} \]

- **Advanced Measurement Approaches (AMA)**
  - Scorecard approach.
  - Loss Distribution approach.
Loss distribution approach

- Model the distribution of the **aggregate losses** for a given **business line & risk type**

\[ \text{Loss}_{t}^{[i,j]} = \sum_{n=1}^{N_t^{[i,j]}} X_{nt}^{[i,j]} ; \]

\[ N_t^{[i,j]} \] is the number of losses in year \( t \) for business line \( i \) and risk type \( j \).

- Calculate **Capital at Risk** (99.9% percentile) of the aggregate loss distribution per business line & risk type and **add** them.

\[ CaR = \sum_{i=1}^{8} \sum_{j=1}^{7} CaR^{[i,j]} \]
Actuarial models: Frequency + Severity

- Hypothesis
  - **Severities** of individual losses are **independent**.
  - **Severities and frequencies** are **independent**.

- Model separately
  - Frequency \( \{N_t\} \)
    - E.g. Poisson, negative binomial, Cox process,…
  - Severity \( \{X_{nt}\} \)
    - E.g. Lognormal, Pareto, g-and-h,…

- Approximate the distribution of aggregate losses by combining these distributions. [Panjer, FFT, MC, single-loss approximations and beyond]
Model distributions for frequencies

- **Poisson model**
  One-parameter model
  average frequency: $\lambda$
  mean = variance

- **Negative binomial**
  Two parameters
  mean < variance
Aggregation of frequency and severity dists.
Expected & unexpected loss
Risk analysis

- Calculate **aggregate yearly loss distribution** from the frequency and severity distributions.

- Compute **risk measures**
  - **Expected loss**
  - **Capital at Risk (CaR)**
    - e.g. 99.9% percentile of the aggregate loss distribution.
  - **Conditional CaR (Expected shortfall)**
    - Expected loss, given that the loss is above CaR
Computational issues in risk analysis

Algorithms to compute **risk measures**

- **Deterministic algorithms**
  - Discretized approximation (FFT, Panjer).
  - Piecewise approximation for the distribution
    - Body approximation: Not very important
    - Tail approximation. E.g. Single loss

- **Monte Carlo algorithms**

  Empirical compound distribution obtained by simulation.

  Computationally costly.

  - Use variance reduction techniques
  - Hardware solutions: *Grid computing, GPU’s, …*
Distribution of aggregate losses

Consider $X_1, \ldots, X_N$ iid samples from the density $f(x)$.

For the individual losses

$$F(x) = \text{Prob}[X \leq x] = \int_0^x f(z) \, dz$$

$$\overline{F}(x) = \text{Prob}[X > x] = 1 - F(x) \quad \text{[Tail distribution]}$$

The aggregate loss $L = X_1 + \ldots + X_N$ is a random variable with density $g(x)$

$$G(x) = \text{Prob}[L \leq x] = \int_0^L g(z) \, dz = F^{*N}(x)$$

$$\overline{G}(x) = \text{Prob}[L > x] = 1 - G(x) \quad \text{[Tail distribution]}$$
The lognormal distribution

\[ LNpdf(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{x} \exp\left\{ -\frac{1}{2\sigma^2} (\log x - \mu)^2 \right\} \]

\( x > 0 \)

\( X \sim \exp(\mu + \sigma Z); \quad Z \sim N(0,1) \)

- \exp(\mu) \Rightarrow \text{scale}
- \sigma \Rightarrow \text{tails}
Sums of iid lognormals

\[ L = X_1 + X_2 + \ldots + X_N \quad \{X_i\} \sim \text{id LN}(\mu, \sigma) \]

- The **sum of lognormals is approximately lognormal** at least for small and intermediate values of \( \sigma \) and large number of terms in the sum.
  - Pricing formulas for **arithmetic-mean Asian options** in terms of corrected formulas for geometric-mean Asian options.
  - Modeling the **level of a stock index** and the prices of its components as lognormal random variables (Black-Scholes)
Sums of iid lognormals

$\mu = 0, \quad \sigma = 1.5, \quad N = 2, 20, 200, 2000$
Sums of iid lognormals

$\mu = 0, \sigma = 1.5, N = 2, 20, 200, 2000$
Sums of iid lognormals

\[ \mu = 0, \quad \sigma = 2.5, \quad N = 2, 20, 200, 2000 \]
Fenton-Wilkinson Approximation

\[ L = X_1 + X_2 + \ldots + X_N \]
\[ \{X_i\} \sim \text{iid LN}(\mu, \sigma) \]

Assumption \( L \sim \text{LN}(\mu_L, \sigma_L) \)

\[
\begin{align*}
\mathbb{E}[L] &= \exp\left( \mu_L + \frac{1}{2} \sigma_L^2 \right) \\
\mathbb{E}[L^2] &= \exp\left( 2\mu_L + \sigma_L^2 \right)
\end{align*}
\]

\[ \begin{cases} 
\mu_L = \log \mathbb{E}[L] - \frac{1}{2} \sigma_L^2 \\
\sigma_L^2 = \log \frac{\mathbb{E}[L^2]}{\mathbb{E}[L]^2}
\end{cases} \]

On the other hand,

\[
\begin{align*}
\mathbb{E}[L] &= N \exp\left( \mu + \frac{1}{2} \sigma^2 \right) \\
\mathbb{E}[L^2] &= \mathbb{E}[L]^2 \left( 1 + \frac{e^{\sigma^2} - 1}{N} \right)
\end{align*}
\]

\[ \begin{cases} 
\mu_L = \mu + \log N + \frac{1}{2} \left( \sigma^2 - \sigma_L^2 \right) \\
\sigma_L^2 = \log \left( 1 + \frac{e^{\sigma^2} - 1}{N} \right)
\end{cases} \]
Fenton-Wilkinson Approximation

<table>
<thead>
<tr>
<th>$\mu_L$</th>
<th>N = 2</th>
<th>N = 20</th>
<th>N = 200</th>
<th>N = 2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0.5$</td>
<td>0.75</td>
<td>3.11</td>
<td>5.42</td>
<td>7.73</td>
</tr>
<tr>
<td>$\sigma = 1.5$</td>
<td>1.11</td>
<td>3.99</td>
<td>6.40</td>
<td>8.72</td>
</tr>
<tr>
<td>$\sigma = 2.5$</td>
<td>1.56</td>
<td>5.41</td>
<td>8.22</td>
<td>10.68</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma_L$</th>
<th>N = 2</th>
<th>N = 20</th>
<th>N = 200</th>
<th>N = 2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0.5$</td>
<td>0.36</td>
<td>0.12</td>
<td>0.04</td>
<td>0.012</td>
</tr>
<tr>
<td>$\sigma = 1.5$</td>
<td>1.17</td>
<td>0.50</td>
<td>0.19</td>
<td>0.065</td>
</tr>
<tr>
<td>$\sigma = 2.5$</td>
<td>1.99</td>
<td>1.04</td>
<td>0.56</td>
<td>0.28</td>
</tr>
</tbody>
</table>

Good results for
- Small $\sigma$
- Very large N

Robust estimation of OR measures
Central limit theorem regime

- Assuming $N$ is sufficiently large so that FW approximation is accurate $[N > \exp(2\sigma^2)]$

  E.g. $\sigma = 2.5 \Rightarrow N > 2.6834\times10^5$

\[
\mu_L = \mu + \log N + \frac{1}{2} \left( \sigma^2 - \sigma_L^2 \right) \approx \log N + \mu + \frac{1}{2} \sigma^2
\]

\[
\sigma_L^2 = \log \left( 1 + \frac{e^{\sigma^2} - 1}{N} \right) \approx \frac{e^{\sigma^2} - 1}{N}
\]

\[
\exp(\mu_L + \sigma_L Z) \approx N \exp\left( \mu + \frac{1}{2} \sigma^2 \right) \left( 1 + \sqrt{\frac{e^{\sigma^2} - 1}{N}} Z \right) \approx \mathbb{E}[L] + \text{stdev}[L] Z;
\]

$Z \sim N(0,1)$
Sums of iid lognormals

\[ L = X_1 + X_2 + \ldots + X_N \]

\[ \{X_i\} \sim \text{iid LN}(\mu, \sigma) \]

For sufficiently large \( \sigma \), and specially at the tails, the sum is dominated by the maximum.

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( N_{\text{central limit}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2</td>
</tr>
<tr>
<td>1.0</td>
<td>8</td>
</tr>
<tr>
<td>1.5</td>
<td>91</td>
</tr>
<tr>
<td>2.0</td>
<td>2981</td>
</tr>
<tr>
<td>2.5</td>
<td>268338</td>
</tr>
<tr>
<td>3.0</td>
<td>65659970</td>
</tr>
<tr>
<td>3.5</td>
<td>4.37E+10</td>
</tr>
<tr>
<td>4.0</td>
<td>7.90E+13</td>
</tr>
<tr>
<td>4.5</td>
<td>3.88E+17</td>
</tr>
<tr>
<td>5.0</td>
<td>5.18E+21</td>
</tr>
</tbody>
</table>

Robust estimation of OR measures
Heavy-tailed distributions

A heavy-tailed distribution is such that the tails of the distribution decay more slowly than the exponential.

$$\lim_{x \to \infty} \frac{F(x)}{e^{-\varepsilon x}} = \infty, \quad \forall \varepsilon > 0$$
Subexponential distributions are a special class of heavy-tailed distributions.

**Definition:** $F(x)$ is a subexponential distribution if

$$
\lim_{x \to \infty} \frac{F^{N*}(x)}{F(x)} = \lim_{x \to \infty} \frac{1 - F^{N*}(x)}{1 - F(x)} = \lim_{x \to \infty} \frac{P(X_1 + \ldots + X_N > x)}{P(X_1 > x)} = N, \quad \forall N \geq 2
$$

where $X_1, \ldots, X_N$ an iid sample from $F(x)$
One loss causes ruin [subexponential distributions]

\[ P(\max(X_1, \ldots, X_N) > x) = 1 - P(\max(X_1, \ldots, X_N) \leq x) \]
\[ = 1 - \prod_{i=1}^{N} P(X_i \leq x) = 1 - [F(x)]^N = (1 - F(x))\sum_{i=0}^{N-1} [F(x)]^i \]
\[ \approx N(1 - F(x)) = N \bar{F}(x) = N P(X_1 > x) \]
\[ \lim_{x \to \infty} \frac{P(X_1 + \ldots + X_N > x)}{P(\max(X_1, \ldots, X_N))} = 1, \quad \forall N \geq 2 \]

One loss causes ruin: If the distribution of individual losses is subexponential, large values of the aggregate loss from N events are dominated by the maximum loss in one of the N events.

Beyond single-loss in operational risk
Single loss approximation [Böcker + Klüppelberg, 2005]

**From the definition of subexponential distribution**

\[ P(X_1 + \ldots + X_N > x) \approx N P(X_1 > x) \quad \text{as} \quad x \to \infty \]

\[ \Rightarrow [1 - P(X_1 + \ldots + X_N \leq x)] \approx N[1 - P(X_1 \leq x)] \]

\[ \Rightarrow [1 - G(x)] \approx N[1 - F(x)] \quad \text{as} \quad x \to \infty \]

**From the definition of \( \text{VaR}_\alpha \)**

\[ P(X_1 > \text{VaR}_\alpha) = 1 - \alpha \quad \Rightarrow \quad P(X_1 \leq \text{VaR}_\alpha) = \alpha \quad \Rightarrow \quad G(\text{VaR}_\alpha) = \alpha \]

\[ 1 - \alpha = N[1 - F(\text{VaR}_\alpha)] \Rightarrow \quad \text{VaR}_\alpha = F^{-1}\left(1 - \frac{1 - \alpha}{N}\right) \]
Single loss approximation + mean correction

**Single loss approximation for the aggregate loss distribution**

[Böcker + Klüppelberg, 2005]

\[
[1 - G(x)] \sim E[N][1 - F(x - (E[N] - 1) \mu_X)] \quad \text{as} \quad x \to \infty
\]

\[
G^{-1}(p) \sim F^{-1}\left(1 - \frac{1 - p}{E[N]}\right) + (E[N] - 1) \mu_X; \quad \mu_X = E[X]
\]

**Single loss approximation + mean correction for \( \text{VaR}_\alpha \)**

\[
\text{VaR}_\alpha = G^{-1}(\alpha) = F^{-1}\left(1 - \frac{1 - \alpha}{E[N]}\right) + (E[N] - 1) \mu_X
\]
Second order asymptotic approximation


\[
[1 - G(x)] \approx E[N][1 - F(x)] \left( 1 - \left( \frac{E[N^2]}{E[N]} - 1 \right) \mu_X \cdot \frac{f(x)}{1 - F(x)} \right) \quad x \to \infty
\]

\[
\Rightarrow G^{-1}(p) \approx F^{-1} \left( 1 - \frac{1 - p}{E[N]} + \left( \frac{E[N^2]}{E[N]} - 1 \right) \mu_X \cdot f\left( G^{-1}(p) \right) \right)
\]

\[
\text{VaR}_\alpha = F^{-1} \left( 1 - \frac{1 - \alpha}{E[N]} + \left( \frac{E[N^2]}{E[N]} - 1 \right) \mu_X \cdot f\left( \text{VaR}_\alpha \right) \right)
\]

Beyond single-loss in operational risk
Iterative algorithm [second order asymptotic approximation]

\[ VaR_\alpha = F^{-1} \left( 1 - \frac{1 - \alpha}{E[N]} + \left( \frac{E[N^2]}{E[N]} - 1 \right) \mu_X f(VaR_\alpha) \right) \]

Iterative algorithm

\[ VaR^{[0]}_\alpha = F^{-1} \left( 1 - \frac{1 - \alpha}{E[N]} \right) \]

\[ VaR^{[k+1]}_\alpha = F^{-1} \left( 1 - \frac{1 - \alpha}{E[N]} + \left( \frac{E[N^2]}{E[N]} - 1 \right) \mu_X f(VaR^{[k]}_\alpha) \right); \quad k = 0,1,2,... \]

Beyond single-loss in operational risk
Second order estimate for Pareto severity

**Poisson**

\[ E[N] = \lambda; \quad E[N^2] = \lambda(\lambda + 1); \quad \frac{E[N^2]}{E[N]} - 1 = \lambda \]

**Pareto**

\[ f(x) = \frac{1}{\xi} \frac{u^{1/\xi}}{x^{1+1/\xi}}; \quad F(x) = 1 - \frac{u^{1/\xi}}{x^{1/\xi}}; \quad F^{-1}(p) = (1 - p)^{-\xi} u \]

\[ \text{VaR}^{[0]}_\alpha = F^{-1} \left( 1 - \frac{1 - \alpha}{\lambda} \right) = \left( \frac{1 - \alpha}{\lambda} \right)^{-\xi} u \]

\[ \text{VaR}^{[1]}_\alpha = F^{-1} \left( 1 - \frac{1 - \alpha}{\lambda} + \lambda \mu x f(\text{VaR}^{[0]}_\alpha) \right) = \text{VaR}^{[0]}_\alpha \left( 1 - \frac{\lambda \mu x}{\xi \text{VaR}^{[0]}_\alpha} \right)^{-\xi} \]
Mean-corrected single-loss approximation

\[
\begin{align*}
V_{\alpha} &= \left( \frac{1 - \alpha}{\lambda} \right)^{-\xi} \\
V_{\alpha} &= V_{\alpha}^{[0]} \left( 1 - \frac{1}{\xi} \frac{\lambda \mu X}{V_{\alpha}^{[0]}} \right)^{-\xi} \\
&= V_{\alpha}^{[0]} \left( 1 + \frac{\lambda \mu X}{V_{\alpha}^{[0]}} + O \left( \frac{\lambda \mu X}{V_{\alpha}^{[0]}} \right)^2 \right)
\end{align*}
\]

\[V_{\alpha}^{[1]} \approx V_{\alpha}^{[0]} + \lambda \mu X \quad \text{[single loss + mean correction]}\]
Second order correction [Degen, 2010]

\[ f(x) = \frac{1}{\xi} \frac{u^{1/\xi}}{x^{1+1/\xi}} \quad \mu_X = \int_x^\infty dx \ x \ f(x) = \frac{1}{1-\xi} u; \quad \text{VaR}_\alpha^{[0]} = \left( \frac{1-\alpha}{\lambda} \right)^{-\xi} u \]

\[ \frac{\text{VaR}_\alpha^{[0]}}{\text{VaR}_\alpha^{[1]}} - 1 \approx K \ A(\alpha) \Rightarrow \]

\[ \text{VaR}_\alpha^{[1]} \approx \text{VaR}_\alpha^{[0]} (1 + K \ A(\alpha))^{-1} \approx \text{VaR}_\alpha^{[0]} (1 - K \ A(\alpha)) \]

\[ -K \ A(\alpha) = \lambda^{1-\xi} \frac{(1-\alpha)^\xi}{1-\xi} u = \lambda \left( \frac{1-\alpha}{\lambda} \right)^\xi \frac{1}{1-\xi} u = \frac{\lambda \mu_X}{\text{VaR}_\alpha^{[0]}} \]

\[ \text{VaR}_\alpha^{[1]} \approx \text{VaR}_\alpha^{[0]} + \lambda \mu_X \]
Accuracy of the correction by the mean

- **Poisson:** $\lambda = 200$
- **Longnormal:** $\sigma = 1.5, 2, 2.5$
Poisson ($\lambda = 200$) + LN ($\mu = 200$)

Beyond single-loss in operational risk
Poisson ($\lambda = 200$) + GP ($u = 2, \theta = 1$)

Beyond single-loss in operational risk
Rapidly varying subexponentials

- The **hazard rate** function is
  \[ h(x) = \frac{f(x)}{F(x)} \Rightarrow F(x) = F(x_0) e^{-\int_{x_0}^{x} h(z) dz} \]

- **Rapidly varying subexponential distribution**
  \[
  \lim_{x \to \infty} x h(x) = \infty; \quad \lim_{x \to \infty} h(x) = 0; \quad \liminf_{x \to \infty} \frac{xh(x)}{\log(x)} > 0
  \]
  - Lognormal distribution \( F(x) \sim \exp\left\{ -(\log x)^a \right\}; \quad a \geq 2 \)
  - Gamma distribution
    \[ f(x; k, \theta) = \frac{1}{\Gamma(k)\theta} \left(\frac{x}{\theta}\right)^{k-1} e^{-x/\theta}; \quad k > 0; \quad \theta > 0 \]
Asymptotic expansion I [Barbe, McCormick & Zhang, 2007]

\[ [1 - G(x)] \approx \lambda \exp \left\{ \lambda \sum_{k=1}^{K} \frac{(-1)^k}{k!} \mu_X^{[k]} \frac{\partial^k}{\partial x^k} \right\} [1 - F(x)] + o(h^K(x)\bar{F}(x)) \]

\( \mu_X^{[k]} = E[X^k] \) \hspace{1cm} \( K = 0, 1, 2, \ldots \)

- **K=0**

\[ [1 - G(x)] \approx \lambda [1 - F(x)] \]

\[ [1 - \alpha] \approx \lambda [1 - F(VaR_0)] \quad \Rightarrow \quad VaR_{\alpha}^{[0]} = F^{-1} \left( 1 - \frac{1 - \alpha}{\lambda} \right) \]

[Single loss approximation]

Beyond single-loss in operational risk
Asymptotic expansion II [Barbe, McCormick & Zhang, 2007]

\[1 - G(x)] \approx \lambda \exp\left\{ \lambda \sum_{n=1}^{m} \frac{(-1)^n}{k!} \mu_X^{[n]} \frac{\partial^n}{\partial x^n} \right\}[1 - F(x)] + o(h^m(x)F(x))

\[\mu_X^{[n]} = E[X^n]\]

\(m = 0, 1, 2, \ldots\)

\(m = 1\)

\[1 - G(x)] \approx \lambda \exp\left\{ -\lambda \mu_X \frac{\partial}{\partial x} \right\}[1 - F(x)] + o(f(x))

\[1 - G(x)] \approx \lambda \left[1 - F(x - \lambda \mu_X)\right]\]

\[1 - \alpha \approx \lambda \left[1 - F(VaR^{[1]}_{\alpha} - \lambda \mu_X)\right] \Rightarrow \text{VaR}^{[1]}_{\alpha} = \text{VaR}^{[0]}_{\alpha} + \lambda \mu_X\]

[Single loss approximation + mean correction]
Second order asymptotic expansion

\[ [1 - G(x)] \approx \lambda \exp \left\{ \frac{1}{2} \lambda \mu_X^{[2]} \frac{\partial^2}{\partial x^2} \right\} [1 - F(x - \lambda \mu_X)] \quad \text{as} \quad x \to \infty \]

\[ [1 - G(x)] \approx \lambda \left[ 1 - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \lambda \mu_X^{[2]}}} \exp \left\{ -\frac{z^2}{2 \lambda \mu_X^{[2]}} \right\} F(z + x - \lambda \mu_X) dz \right] \]
Asymptotics with a shifted argument
[Albrecher, Hipp & Kortschak, 2011]

Consider the expansion
\[
[1 - G(x)] = E[N][1 - F(x)] + \sum_{n=1}^{m} \frac{1}{n!} E[N(X_1 + \ldots + X_{N-1})^n] D_x^n F(x) + o(D_x^m F(x))
\]

The corresponding expansion with a shifted argument is
\[
[1 - G(x)] \approx x_1 E[N][1 - F(x - k_1)] + \ldots + x_m E[N][1 - F(x - k_m)]
\]

Taking \(m = 1\)
\[
x_1 E[N][1 - F(x - k_1)] \approx E[N][1 - F(x)] + E[N(X_1 + \ldots + X_{N-1})] D_x F(x)
\]

\[\Rightarrow x_1 = 1\]

\[\Rightarrow k_1 = \frac{E[N(X_1 + \ldots + X_{N-1})]}{E[N]} = \frac{E[N(N-1)]}{E[N]} E[X] = \lambda \mu_x\]
Perturbative expansion [Hernández & Tejero, 2011]

\[ \text{VaR}_\alpha = Q_0 + \varepsilon Q_1 + \frac{1}{2} \varepsilon^2 Q_2 + \ldots \]

with

\[ Q_0 = F^{-1}(\alpha^{1/N}), \quad Q_1 = (N - 1)E[X | X \leq Q_0] \]

Using the first order approximation and assuming that \( Q_0 >> \mu_X \)

\[ \text{VaR}_\alpha^{[0]} = F^{-1}(\alpha^{1/N}) \approx F^{-1}\left(1 - \frac{1 - \alpha}{E[N]}\right) \]

\[ \text{VaR}_\alpha^{[1]} = \text{VaR}_\alpha^{[0]} + (N - 1)E[X | X \leq Q_0] \approx \text{VaR}_\alpha^{[0]} + (N - 1)\mu_X \]
Accuracy of the perturbative scheme

- Poisson: $\lambda = 20$
- Longnormal: $\sigma = 1.5, 2, 2.5$
Accuracy of the perturbative scheme

- Poisson: \( \lambda = 200 \)
- Longnormal: \( \sigma = 1.5, 2, 2.5 \)
Estimation of the mean can be difficult

Beyond single-loss in operational risk
Error in the sample estimate of the mean

Beyond single-loss in operational risk
No CLT in the convergence to the mean

\[ \mu = 0; \sigma = 3 \quad \text{mean} = 90.0171 \]
Local tail index

For the Pareto distribution

\[ P(X > x) = \left( \frac{u}{x} \right)^{1/\xi} \Rightarrow \mu_X = \frac{u}{1-\xi}; \quad P(X > n\mu_X) = \left( \frac{1-\xi}{n} \right)^{1/\xi}; \quad 0 < \xi < 1; \]

For an arbitrary distribution

\[ \left( \frac{1-\xi(n)}{n} \right)^{1/\xi(n)} \equiv P(X > n\mu_X) \]
Asymptotic formulas to operational VaR

- The **single-loss** approximation is insufficient, specially with:
  - Lower percentiles.
  - Less heavy tails.
  - Higher frequencies.
- The **second order asymptotic approximation**
  - Improves estimate and is easy to compute.
- The **single loss formula corrected by the mean**
  - Can be derived from the second order asymptotics.
  - Accurate in a wide range of cases.
  - Estimates of the mean can be very inaccurate if tails are heavy.
White swans

- Exponential (light) tails
- Typical
- Bounded impact
- Can be modeled and predicted
Black Swans [N. N. Taleb]

- Rare events
- Extreme impact
- Retrospective but not prospective predictability
- Cannot be modeled (only option: robustification)

Beyond single-loss in operational risk
Grey swans

- Rare events
- Extreme impact
- Retrospective but not prospective predictability
- Subexponential (heavy) tails

I exist!
I can be modeled!