# Drift conditions on a HJM model with stochastic basis spreads

Teresa Martínez
Quantitative Product Group

Santander Quantitative Product Group

# Contents

**Abstract.** A general HJM model is considered to model the evolution of discount and reference curves (in several currencies) under the risk-neutral probability associated to the discount curve of the domestic economy. Drift conditions are derived for these dynamics that make the model consistent with the market practice of curve construction. Some numerical tests are included in the particular case of the Hull-White model.

# 1 Introduction

The global economy has been suffering, since subprime mortgage crisis arose in summer 2007, important modifications in markets all over the world. Not only the volume of operations, affected by liquidity and credit constraints, has experienced this impact. The whole understanding of the trading machinery has gone under review as a byproduct of such changing situation. For fixed income securities, one of the consequences of the credit crunch, among others, is the widening of spreads between formerly equivalent rates (or with negligible differences for curve construction purposes) for instruments of the same maturity but different underlying tenors. This divergence originates in the changes of credit and liquidity disposal among the market agents. A heuristical explanation of this effect can be found in [M].

To our knowledge, there is no general theory already accepted to rigorously explain this effects from the theoretical point of view. Nevertheless, the market agents have already developed a way to treat this divergence, becoming a usual market practice the valuation of fixed income derivatives by using different curves: a unique curve to discount flows, and different ones to estimate floating flows. In the classical one–curve setting, the valuation of the floating payment of a standard market FRA is given by a replication strategy consistent in selling one bond and buying another one. In this setting, the floating payment can be expressed as

$$FRA_{floating}(0,T,U) = B(0,T) - B(0,U) = m(T,U)B(0,U)\left[\left(\frac{B(0,T)}{B(0,U)} - 1\right)\frac{1}{m(T,U)}\right], \tag{1.1}$$

where $B(0,t)$ stands for the price today of a bond paying one unit of currency at time $t$ (the discount factor at time $t$), $\left(\frac{B(0,T)}{B(0,U)} - 1\right)\frac{1}{m(T,U)}$ is the expression of the Libor rate prevailing today for the accrual period $(T,U)$ and $m(T,U)$ is the day count fraction from $T$ to $U$ according to market convention. This expression is derived, from the definition of a FRA, through the use of well known no–arbitrage arguments. These arguments replicate market behavior as long as the market admits this replication strategy. Actually, this is not the case, and the forward rates implied from the deposit rates (equivalent to the expression appearing in (1.1)) are not necessarily equal to the quoted FRA rates. These are then replicated by using a different curve, in such a way that the quoted FRA rate corresponds to

$$\left(\frac{B_r(0,T)}{B_r(0,U)} - 1\right)\frac{1}{m(T,U)},$$

where $B_r(0,t)$ is the discount factor calculated according to a special curve, built for rates of tenor $U-T$ (typically $U-T = 1M$, $3M$, $6M$ and $12M$ ). Then, the price of a FRA is, in this two curve setting,

$$FRA_{floating}(0,T,U) = m(T,U)B_d(0,U)\left[\left(\frac{B_r(0,T)}{B_r(0,U)} - 1\right)\frac{1}{m(T,U)}\right], \tag{1.2}$$

Any model that intends to fit this situation needs to cope with the fact that the usual no–arbitrage arguments have to be carefully applied.

In the present work, we consider the problem of defining a HJM model consistent with this market practice. More concretely, we assume that under the discount bank account risk-free probability, $\mathbb{P}_d$ with associated numeraire $e^{\int_0^t r_d(s)\,ds}$, the instantaneous forward rates (**i.f.r.** from now on) of the discount and reference curves, $f_d(t,T)$ and $f_r(t,T)$, respectively, follow the dynamics

$$\begin{aligned} df_d(t,T) &= \mu_d(t,T)dt + \sigma_d(t,T)dW_t^d, \quad r_d(t) = f_d(t,t) \\ df_r(t,T) &= \mu_r(t,T)dt + \sigma_r(t,T)dW_t^r, \quad r_r(t) = f_r(t,t). \end{aligned}$$

By classical no-arbitrage theory, it is known that

$$\mu_d(t,T) = \sigma_d(t,T)\int_t^T \sigma_d(t,s)ds. \tag{1.3}$$

Our aim is finding the expression for the drift of the reference curve, in such a way that

$$FRA_{floating}(t,T,U) = E_t^{\mathbb{P}}\left(e^{-\int_t^U r_d(s)\,ds}L_r(T^*,T,U)\right) = B_d(t,U)L_r(t,T,U), \tag{1.4}$$

where $L_r(T^*,T,U)$ is the Libor fixing at $T^*$ with accrual period $(T,U)$, whose expression we assume to be

$$L_r(t,T,U) = \left(\frac{B_r(t,T)}{B_r(t,U)} - 1\right)\frac{1}{m(T,U)} = \left(e^{\int_T^U f_r(t,T')\,dT'} - 1\right)\frac{1}{m(T,U)}.$$

Observe that this requirement is a natural extension of the market definition of reference curves, that translates into expressions like (1.2). Of course, (1.4) is stated for each reference curve, only for floating payments of the Libor index associated to the curve. Also, we consider this problem in the multi–currency setting, giving the expressions of the drift of the foreign discount and reference curves under the domestic discounting risk-neutral probability.

It is convenient to note that we are not dealing with the construction of reference curves, something that constitutes a research topic in itself. We assume the existence of a well defined set of curves, with appropriate interpolation and bootstrapping algorithms, and only give conditions on the diffusions of the i.f.r.'s dynamics to meet condition (1.4).

As far as we know, this approach to the two curves valuation in the HJM setting is new. There are precedents of HJM models extensions to valuate credit spreads, by postulating the evolution of the defaultable bonds as given by a defaultable i.f.r.with an HJM dynamics. See, for example, [DS], [S2] and [ChFM] (see also [S1] for a similar approach for a Libor Market Model). In all these references, the modeling of the (defaultable) rates is ruled by credit constraints. Al though the origin of the existence of reference curves is the already mentioned credit and liquidity crisis, it is not clear how to incorporate it in the diffusion of reference rates by using a similar approach. In [JP] and [AS], general frameworks to incorporate liquidity risk in the valuation of market securities are studied. They extend classical no–arbitrage pricing theory to an economy in which the securities price depends also on the size of the trade considered.

Basis spreads among rates for instruments of the same maturity denominated in the same currency have been considered negligible until summer 2007 credit crisis. Before this point, basis spreads for rates have been mainly studied in the multi–currency case, to price cross currency derivatives, see for instance [BS] and [T]. The appearance of basis spreads among one–currency market rates is treated in [Bi] by using an FX analogy. This methodology is used to derive prices of vanilla products (cap/floor, swaps and swaptions) in a one–currency multi–curve setting, and produces valuation formulae with quanto–like adjustments. Unadjusted formulae are derived in [M] in an extension of Libor Market Models to the multi–curve setting. In [KTW], a three curve model (with one-factor quadratic Gaussian dynamics) is proposed, to study option prices on swaps and bonds.

The organization of the paper is as follows. The notation to be used throughout the paper is stated in section 2, in section 3 the main results are stated, and in section 4, an application to a Hull-White model is given.

## 2 Setting of the problem. Notation

Our aim is being able to derive a multi–curve multi–currency HJM model in a consistent way with the market practice. To this aim, we consider a domestic economy ruled by a discount curve and several reference curves, such that floating payments depend on rates related to the reference curves, and are discounted with the discount reference curve. The i.f.r. for the reference curve can be expressed as the sum of the discounting i.f.r. plus a spread i.f.r., that we will also consider in the model. In the foreign economy, a similar situation happens, and we have a FX spot process to change from one foreign economy to the domestic one. For simplicity, we will consider only a reference curve in the domestic economy, only one foreign economy, and only one reference curve in the foreign economy (all the results would apply in the general case, simply repeating the arguments with each domestic reference curve and each foreign economy). In this situation, we will consider the following notation for the different i.f.r.:

| | |
|---|---|
| $f_d^d(t,T)$ | discounting i.f.r.in the domestic economy |
| $f_d^f(t,T)$ | discounting i.f.r. in the foreign economy |
| $f_r^d(t,T)$ | reference i.f.r. in the domestic economy |
| $f_r^f(t,T)$ | reference i.f.r. in the foreign economy |
| $f_s^d(t,T)$ | spread i.f.r. in the domestic economy $f_r^d(t,T) = f_d^d(t,T) + f_s^d(t,T)$ |
| $f_s^f(t,T)$ | spread i.f.r. in the foreign economy $f_r^f(t,T) = f_d^f(t,T) + f_s^f(t,T)$ |

All of these rates live and evolve in a filtered space $(\Omega, \mathcal{F}, \mathbb{P}_d, \{\mathcal{F}_t\})$, rich enough to carry a $\mathbb{P}_d$–Brownian motion of the suitable dimension, where $\mathbb{P}_d$ is the domestic risk–neutral probability. We will consider that each of these rates satisfy an Itô diffusion,

$$df_\gamma^\alpha(t,T) = \mu_\gamma^\alpha(t,T)\,dt + \sigma_\gamma^\alpha(t,T)dW_t^{\alpha\gamma}$$

with $\alpha = d, f$ (domestic, foreign) denoting the economy we are referring to, $\gamma = d, r, s$ (discount, reference, spread) denoting the curve in each economy and $W_t^{\alpha\gamma}$ a Brownian motion under the domestic risk–neutral probability. Also, we write

$$\rho_{\alpha'\gamma'}^{\alpha\gamma}(t)\,dt = d\langle W^{\alpha\gamma}, W^{\alpha'\gamma'}\rangle_t$$

the instantaneous correlation between any two of the Brownian motions driving i.f.r.'s diffusions. We assume that all the processes depending on a maturity parameter $T$ are smooth enough to be differentiable and integrable w.r.t this parameter, and Fubini's theorem holds.

## 2.1 Bonds and curves

We will often use that under each curve, the "bond price" process is given by

$$B_\gamma^\alpha(t,T) = e^{-\int_t^T f_\gamma^\alpha(t,s)\,ds}, \quad \alpha = d, f, \;\; \gamma = d, r, s. \tag{2.1}$$

Let us forget for a moment the super and subscripts $\alpha$, $\gamma$, since the following calculation is valid in any case, and define

$$Y(t,T) = \int_t^T f(t,s)ds,$$

such that the bond price can also be computed as $B(t,T) = e^{-Y(t,T)}$. For this process, we have

$$\begin{aligned} Y(t,T) &= \int_t^T [f(0,s) + \int_0^t \mu(u,s)\,du + \int_0^t \sigma(u,s)\,dW_u]\,ds \\ &= \int_t^T f(0,s)\,ds + \int_0^T \int_0^t \mu(u,s)\,du\,ds + \int_0^T \int_0^t \sigma(u,s)\,dW_u\,ds \\ &\quad - \int_0^t \int_0^t \mu(u,s)\,du\,ds - \int_0^t \int_0^t \sigma(u,s)\,dW_u\,ds. \end{aligned}$$

Note that

$$\begin{aligned}\int_0^T \int_0^t \mu(u,s)\,du\,ds &= \int_0^t \int_0^T \mu(u,s)\,ds\,du \\ &= \int_0^t \int_0^u \mu(u,s)\,ds\,du + \int_0^t \int_u^T \mu(u,s)\,ds\,du,\end{aligned}$$

and the same happens for the other integrals. By substituting these expressions in the former equation, and simplifying the repeated terms, we get

$$\begin{aligned}Y(t,T) &= \int_0^T f(0,s)\,ds + \int_0^t \int_u^T \mu(u,s)\,ds\,du + \int_0^t \int_u^T \sigma(u,s)\,ds\,dW_u \\ &\quad - \int_0^t f(0,s)\,ds - \int_0^t \int_u^t \mu(u,s)\,ds\,du - \int_0^t \int_u^t \sigma(u,s)\,ds\,dW_u.\end{aligned}$$

By changing the order of integration, we get

$$\int_0^t \int_u^t \mu(u,s)\,ds\,du = \int_0^t \int_0^s \mu(u,s)\,du\,ds$$
$$\int_0^t \int_u^t \sigma(u,s)\,ds\,dW_u = \int_0^t \int_0^s \sigma(u,s)\,dW_u\,ds.$$

By observing that

$$f(t,t) = f(0,t) + \int_0^t \mu(s,t)\,ds + \int_0^t \sigma(s,t)\,dW_s$$

and defining:

$$A(t,T) = \int_t^T \mu(t,s)ds, \quad S(t,T) = \int_t^T \sigma(t,s)ds, \tag{2.2}$$

we get

$$Y(t,T) = \int_0^T f(0,s)\,ds + \int_0^t A(u,T)du + \int_0^t S(u,T)dW_u - \int_0^t f(s,s)ds.$$

Finally, we have

$$\begin{aligned}B(t,T) &= e^{-Y(t,T)} \\ &= \exp\left\{ - \int_0^T f(0,s)\,ds - \int_0^t A(u,T)du - \int_0^t S(u,T)dW_u + \int_0^t f(s,s)ds.\right\}. \end{aligned} \tag{2.3}$$

And by Itô's rule,

$$\frac{dB(t,T)}{B(t,T)} = \left(f(t,t) - A(t,T) + \frac{1}{2}(S(t,T))^2\right) dt - S(t,T)\,dW_t \tag{2.4}$$

Another useful expression is the dynamics of

$$\Phi_t = B_1(t,S)\frac{B_2(t,T)}{B_2(t,U)}$$

with $B_1$ and $B_2$ defined as in (2.1) for two different i.f.r., $f_1$ and $f_2$. In this situation, we can write

$$\Phi_t = \Phi_0 e^{\phi_t}$$

where

$$\begin{aligned}\phi_t &= -\int_0^t A_1(u,S)du - \int_0^t S_1(u,S)dW_u^1 + \int_0^t f_1(s,s)ds + \\ &\quad + \int_0^t \left(A_2(u,U) - A_2(u,T)\right)du + \int_0^t (S_2(u,U) - S_2(u,T))dW_u^2.\end{aligned}$$

Another application of Itô's rule gives

$$\begin{aligned}\frac{d\Phi_t}{\Phi_t} &= d\phi_t + \frac{1}{2}d\langle\phi\rangle_t \\ &= \Big(f_1(t,t) - A_1(t,S) + A_2(t,U) - A_2(t,T) \\ &\quad + \frac{1}{2}(S_1(t,S))^2 + \frac{1}{2}(S_2(t,U) - S_2(t,T))^2 - \rho_2^1(t)S_1(t,S)(S_2(t,U) - S_2(t,T))\Big)\,dt \\ &\quad - S_1(t,S)dW_t^1 + (S_2(t,U) - S_2(t,T))dW_t^2.\end{aligned} \tag{2.5}$$

### 2.2 The FX spot process

The FX spot processes $S_t$ is defined as the number of units of domestic currency per one unit of foreign currency, satisfying

$$\frac{dS_t}{S_t} = (f_d^d(t,t) - f_d^f(t,t))\,dt + \sigma^S(t)\,dW_t^S, \tag{2.6}$$

with

$$\rho_S^{\alpha\gamma}(t) = d\langle W^{\alpha\gamma}, W^S\rangle_t$$

the instantaneous correlation between the Brownian motions of the i.f.r. and FX spot. In our calculations, only $\rho_S^{fd}$ will appear.

## 3 Drifts of the curves

### 3.1 Domestic currency discount curve i.f.r. drift: the HJM condition

Throughout this section we consider

$$f_d^d(t,T) := f(t,T) = f(0,T) + \int_0^t \mu(s,T)\,ds + \int_0^t \sigma(s,T)dW_s.$$

By (2.4), in order to avoid arbitrage we must impose:

$$A(t,T) = \frac{1}{2}S^2(t,T). \tag{3.1}$$

Differentiating both sides with respect to $T$, the former condition is equivalent to require the domestic discounting i.f.r. drift satisfies:

$$\mu(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds. \tag{3.2}$$

### 3.2 Domestic currency estimation curve

We consider a model for domestic floating payments ruled by two curves, a discount curve and a reference curve, whose i.f.r. are given, respectively, by:

$$\begin{aligned} df_d^d(t,T) &:= df_d\,(t,T) = \mu_d\,(t,T)\,dt + \sigma_d\,(t,T)\,dW_t^d, \\ df_r^d(t,T) &:= df_r\,(t,T) = \mu_r\,(t,T)\,dt + \sigma_r\,(t,T)\,dW_t^r, \end{aligned}$$

with

$$d\langle W^d, W^r\rangle_t = \rho_r^d(t)\,dt.$$

The floating payments considered are such that the discount of the flow is calculated with the discount curve, and the applicable Libor rate is calculated with the reference curve. We will also consider the formulation of the problem in terms of the spread curve, whose i.f.r. is built as the difference between the discount and the reference i.f.r. s, namely

$$f_s(t,T) = f_d(t,T) - f_r(t,T).$$

We denote by

$$L_r(T^*,T,U) = \left(\frac{B_r(T^*,T)}{B_r(T^*,U)} - 1\right)\frac{1}{m\,(T,U)} = \left(e^{\int_T^U f_r(T^*,T')\,dT'} - 1\right)\frac{1}{m\,(T,U)}$$

the Libor rate fixing at $T^*$, with natural accrual period $(T,U)$.

Our aim is finding a suitable expression for the drift of the reference (or spread) i.f.r. such that the expected value at any time $t \le T^*$ of the discounted floating flow is given by (1.4). Since we can write

$$E_t^{\mathbb{P}}\left(e^{-\int_t^U f_d(s,s)\,ds} L_r(T^*,T,U)\right) = E_t^{\mathbb{P}}\left(e^{-\int_t^{T^*} f_d(s,s)\,ds} B_d(T^*,U) L_r(T^*,T,U)\right),$$

condition (1.4) is satisfied if the process

$$X_u = e^{-\int_t^u f_d(s,s)\,ds} B_d(u,U) L_r(u,T,U)$$

is martingale under the risk neutral measure. This will be our assumption.

Note that this condition is immediately satisfied, by the usual no–arbitrage theory, if $B_d(u,U)L_r(u,T,U)$ represents the price of a tradable asset[1]. But we are not postulating this, instead, we are only imposing a condition on the dynamics of the process.

In fact, it is enough if we require the process

$$Z_t = B_d(t,U)\frac{B_r(t,T)}{B_r(t,U)} = B_d(t,T)\frac{B_s(t,T)}{B_s(t,U)} \tag{3.3}$$

to satisfy an Itô diffusion of the form

$$\frac{dZ_t}{Z_t} = f_d(t,t)\,dt + (\text{a martingale term}).$$

[1] At time $u$, $L_r(u,T,U)$ is known, and the quantity $B_d(u,U)L_r(u,T,U)$ is a fixed number of contracts paying a unit of currency at time $U$.

By (2.5) with $f_1 = f_d$, $f_2 = f_r$, $S = U$, we have

$$\begin{aligned} \frac{dZ_t}{Z_t} &= f_d(t,t)\,dt + \big(A_r(t,U) - A_r(t,T) - A_d(t,U)\big)\,dt \\ &+ \frac{1}{2}\Big[(S_r(t,U) - S_r(t,T))^2 + (S_d(t,U))^2 - 2\rho_s^r(t)S_d(t,U)(S_r(t,U) - S_r(t,T))\Big]\,dt \\ &+ (S_s(t,U) - S_s(t,T))\,dW_t^s - S_d(t,U)dW_t^d. \end{aligned} \tag{3.4}$$

Condition (1.4), by using (3.2), means that we are imposing

$$\begin{aligned} \int_T^U \mu_r(t,t')\,dt' &= -\frac{1}{2}(S_r(t,U) - S_r(t,T))^2 + \rho_r^d(t)S_d(t,U)(S_r(t,U) - S_r(t,T)) \\ &= -\int_T^U \sigma_r(t,t') \int_T^{t'} \sigma_r(t,t'')\,dt''\,dt' + \rho_r^d(t)\int_t^T \sigma_d(t,t')\,dt' \int_T^U \sigma_r(t,t')\,dt'. \end{aligned} \tag{3.5}$$

We write $U = T + \delta$ ($\delta$ the tenor of the Libor index associated to the reference curve), and assume (3.5) holds for any $T \geq t$

$$\int_T^{T+\delta} \mu_r(t,t')\,dt' = g_r(t,T),$$

with

$$g_r(t,T) = -\int_T^{T+\delta} \sigma_r(t,t') \int_T^{t'} \sigma_r(t,t'')\,dt''\,dt' + \rho_r^d(t)\int_t^{T+\delta} \sigma_d(t,t')\,dt' \int_T^{T+\delta} \sigma_r(t,t')\,dt',$$

for any $T \geq t$. In particular, choosing $T = t$,

$$g_r(t,t) = -\int_t^{t+\delta} \sigma_r(t,t') \int_t^{t'} \sigma_r(t,t'')\,dt''\,dt' + \rho_r^d(t)\int_t^{t+\delta} \sigma_d(t,t')\,dt' \int_t^{t+\delta} \sigma_r(t,t')\,dt',$$

and there are an infinity of possible choices for the value of $\mu_r(t,t')$ with $t' \in [t, t+\delta)$ satisfying

$$\int_t^{t+\delta} \mu_r(t,t')\,dt' = g_r(t,t),$$

for example

$$\mu_r(t,t') = \frac{1}{\delta} g_r(t,t).$$

or

$$\mu_r(t,t') = -\sigma_r(t,t')\int_t^{t'} \sigma_r(t,t'')\,dt'' + \rho_r^d(t)\sigma_r(t,t')\int_t^{t+\delta} \sigma_d(t,t')\,dt'.$$

or

$$\mu_r(t,t') = -\sigma_r(t,t')\int_t^{t'} \sigma_r(t,t'')\,dt'' + \rho_r^d(t)\sigma_d(t,t')\int_t^{t+\delta} \sigma_r(t,t')\,dt'.$$

For $T \geq t + \delta$, we can differentiate in $T$ our condition to get that $\mu_r$ should satisfy

$$\mu_r(t, T+\delta) - \mu_r(t,T) = \frac{\partial g_r}{\partial T}(t,T),$$

with

$$\frac{\partial g_r}{\partial T}(t,T) = -(\sigma_r(t,T+\delta)-\sigma_r(t,T))\int_T^{T+\delta}\sigma_r(t,t'')\,dt''\,dt' + \rho_r^d(t)\sigma_d(t,T+\delta)\int_T^{T+\delta}\sigma_r(t,t')\,dt' + \rho_r^d(t)\int_t^{T+\delta}\sigma_d(t,t')\,dt'(\sigma_r(t,T+\delta)-\sigma_r(t,T)).$$

For $T\in[t+i\delta,t+(i+1)\delta)$, we can write

$$\begin{aligned}\mu_r(t,T) &= \mu_r(t,T-\delta)+\frac{\partial g_r}{\partial T}(t,T-\delta) = \mu_r(t,T-2\delta)+\frac{\partial g_r}{\partial T}(t,T-\delta)+\frac{\partial g_r}{\partial T}(t,T-2\delta) = \ldots\\ &= \mu_r(t,T-i\delta)+\sum_{j=1}^{i}\frac{\partial g_r}{\partial T}(t,T-j\delta).\end{aligned}$$

With this expression, we easily have that condition (3.5) is satisfied for any choice of the expression of $\mu_r(t,T)$ when $T$ is in $[t,t+\delta)$.

### 3.2.1 In terms of spread curve i.f.r. drift

By (2.5) with $f_1=f_d$, $f_2=f_s$, $S=T$, we have that the process (3.3) satisfies

$$\begin{aligned}\frac{dZ_t}{Z_t} &= f_d(t,t)\,dt+\big(A_s(t,U)-A_s(t,T)-A_d(t,T)\big)\,dt\\ &+\frac{1}{2}\Big[(S_s(t,U)-S_s(t,T))^2+(S_d(t,T))^2-2\rho_s^d(t)S_d(t,T)(S_s(t,U)-S_s(t,T))\Big]\,dt\\ &+(S_s(t,U)-S_s(t,T))\,dW_t^s-S_d(t,T)dW_t^d.\end{aligned}\tag{3.6}$$

Condition (1.4), by using (3.2), means that we are imposing

$$\begin{aligned}\int_T^U\mu_s(t,t')\,dt' &= -\frac{1}{2}(S_s(t,U)-S_s(t,T))^2+\rho_s^d(t)S_d(t,T)(S_s(t,U)-S_s(t,T))\\ &= -\int_T^U\sigma_s(t,t')\int_T^{t'}\sigma_s(t,t'')\,dt''\,dt'+\rho_s^d(t)\int_t^T\sigma_d(t,t')\,dt'\int_T^U\sigma_s(t,t')\,dt'.\end{aligned}\tag{3.7}$$

We write $U=T+\delta$ with $\delta$ the tenor of the index considered a fixed amount of time, and assume (3.7)

$$\int_T^{T+\delta}\mu_s(t,t')\,dt' = g_s(t,T),$$

with

$$g_s(t,T) = -\int_T^{T+\delta}\sigma_s(t,t')\int_T^{t'}\sigma_s(t,t'')\,dt''\,dt'+\rho_s^d(t)\int_t^T\sigma_d(t,t')\,dt'\int_T^{T+\delta}\sigma_s(t,t')\,dt',$$

for any $T\geq t$. In particular, choosing $T=t$,

$$g_s(t,t) = -\int_t^{t+\delta}\sigma_s(t,t')\int_t^{t'}\sigma_s(t,t'')\,dt''\,dt',$$

and there are an infinity of possible choices for the value of $\mu_s(t,t')$ with $t' \in [t, t+\delta)$ satisfying

$$\int_t^{t+\delta} \mu_s(t,t')\, dt' = g_s(t,t),$$

for example

$$\begin{aligned} \mu_s(t,t') &= -\sigma_s(t,t') \int_t^{t'} \sigma_s(t,t'')\, dt'', \quad \text{or} \\ \mu_s(t,t') &= -\frac{1}{\delta} \int_t^{t+\delta} \sigma_s(t,a) \int_t^{a} \sigma_s(t,t'')\, dt''\, da, \quad \text{etc.} \end{aligned}$$

For $T \geq t+\delta$, we can differentiate in $T$ our condition to get that $\mu_s$ should satisfy

$$\mu_s(t, T+\delta) - \mu_s(t,T) = \frac{\partial g_s}{\partial T}(t,T),$$

with

$$\begin{aligned} \frac{\partial g_s}{\partial T}(t,T) &= -(\sigma_s(t,T+\delta) - \sigma_s(t,T)) \int_T^{T+\delta} \sigma_s(t,t'')\, dt'' \\ &\quad + \rho_s^d(t)\sigma_d(t,T) \int_T^{T+\delta} \sigma_s(t,t')\, dt' + \rho_s^d(t) \int_t^{T} \sigma_d(t,t')\, dt' (\sigma_s(t,T+\delta) - \sigma_s(t,T)). \end{aligned}$$

For $T \in [t+i\delta, t+(i+1)\delta)$, we can write

$$\begin{aligned} \mu_s(t,T) &= \mu_s(t,T-\delta) + \frac{\partial g_s}{\partial T}(t,T) = \mu_s(t,T-2\delta) + \frac{\partial g_s}{\partial T}(t,T-2\delta) + \frac{\partial g_s}{\partial T}(t,T-\delta) = ... \\ &= \mu_s(t,T-i\delta) + \sum_{j=1}^{i} \frac{\partial g_s}{\partial T}(t,T-j\delta). \end{aligned}$$

With this expression, we easily have that condition (3.7) is satisfied for any choice of the expression of $\mu_s(t,T)$ when $T$ is in $[t, t+\delta)$.

### 3.3 Foreign currency discount curve i.f.r. drift

For each foreign economy considered in our model, we assume the following diffusions for the domestic and foreign discounting i.f.r. and for the FX spot:

$$df_d^d(t,T) := df^d(t,T) = \mu^d(t,T)dt + \sigma^d(t,T)dW_t^d, \tag{3.8}$$

$$df_d^f(t,T) := df^f(t,T) = \mu^f(t,T)dt + \sigma^f(t,T)dW_t^f, \tag{3.9}$$

$$\frac{dS(t)}{S(t)} = [f_d^d(t,t) - f_d^f(t,t)]dt + \sigma^S(t)dW_t^S. \tag{3.10}$$

Consider the quantity $S(t)B^f(t,T)$. Let us recall that in this work, we are not discussing the construction of the discount and estimation curves. We consider that foreign curves are constructed, for example, by using the methodology proposed in [BS], and that $f_d^f(t,t)$ is the spot curve used under $\mathbb{P}_d$ measure to discount all flows in the foreign economy. Thus,

$$B^f(0,T) = e^{-\int_0^T f^f(0,s)\, ds}$$

would be the price today (in units of foreign currency) of a contract paying one unit of foreign currency at time $T$, seen from the domestic economy. Typically, this price would include the cross currency basis risk as shown in [BS]. Therefore, naturally extending this notion to time $t > 0$, we see that $B^f(t,T)$ represents the price at time $t$ of a contract paying a unit of foreign currency at time $T$, expressed in foreign units of currency, seen from the domestic economy. Multiplying it by $S(t)$ transforms it into units of domestic currency. The quantity $S(t)B^f(t,T)$ is therefore the price of a tradable asset in the domestic economy. Applying Îto's lemma to the process

$$X_t = S(t)B^f(t,T),$$

and by using (3.10) and (2.4), we get

$$\begin{aligned}\frac{dX_t}{X_t} &= (f^d(t,t) - f^f(t,t))\,dt + (f^f(t,t) - A^f(t,T) + \frac{1}{2}(S^f(t,T))^2)\,dt \\ &\quad -\rho^f_S(t)\sigma^S(t)S^f(t,T)\,dt + \sigma^S(t)\,dW^S_t - S^f(t,T)\,dW^f_t,\end{aligned}$$

with $A^f$, $S^f$ as in (2.2). By no arbitrage arguments, the drift in this Itô equation must be $f^d(t,t)$, and therefore we impose

$$A^f(t,T) = \frac{1}{2}(S^f(t,T))^2 - \rho^f_S(t)S^f(t,T)\sigma^S(t). \tag{3.11}$$

Differentiating in $T$, this equation gives

$$\mu^f(t,T) = \sigma^f(t,T)\int_t^T \sigma^f(t,s)ds - \rho^f_S\sigma^f(t,T)\sigma^S(t). \tag{3.12}$$

### 3.4 Foreign currency estimation curve

For each foreign economy considered in our model, we assume the following diffusions for the domestic and foreign discounting i.f.r., foreign reference i.f.r. and for the FX spot:

$$df^d_d(t,T) := \mu^d_d(t,T)dt + \sigma^d_d(t,T)dW^{dd}_t, \tag{3.13}$$

$$df^f_d(t,T) := \mu^f_d(t,T)dt + \sigma^f_d(t,T)dW^{fd}_t, \tag{3.14}$$

$$df^f_r(t,T) := \mu^f_r(t,T)dt + \sigma^f_r(t,T)dW^{fr}_t, \tag{3.15}$$

$$\frac{dS(t)}{S(t)} = [f^d_d(t,t) - f^f_d(t,t)]dt + \sigma^S(t)dW^S_t. \tag{3.16}$$

In the former section, the condition to be satisfied was (1.4). In the case of the foreign curves, we will require the same condition, that is,

$$FRA^f_{floating}(t,T,U) = E^{\mathbb{P}}_t\left(e^{-\int_t^U r^f_d(s)\,ds}L^f_r(T^*,T,U)\right) = B^f_d(t,U)L^f_r(t,T,U),$$

since the curves are constructed to satisfy this condition at $t = 0$. Observe that this condition is completely analogous to the one imposed for the domestic reference curve with respect to the domestic discount curve. Following the same reasoning as in the domestic case (observe that no arbitrage arguments were used), we get that the drifts of the reference curve (alternatively, spread curve) are as follows. For $T \in [t+i\delta, t+(i+1)\delta)$, $i \geq 1$

$$\mu_r^f(t,T) = \mu_r^f(t,T-i\delta) + \sum_{j=1}^{i} \frac{\partial g_r^f}{\partial T}(t,T-j\delta),$$

with

$$\begin{aligned}\frac{\partial g_r^f}{\partial T}(t,T) &= -(\sigma_r^f(t,T+\delta) - \sigma_r^f(t,T)) \int_T^{T+\delta} \sigma_r^f(t,t'')\,dt''\,dt' \\ &\quad + \rho_{fr}^{fd}(t)\sigma_d^f(t,T+\delta) \int_T^{T+\delta} \sigma_r^f(t,t')\,dt' + \rho_{fr}^{fd}(t) \int_t^{T+\delta} \sigma_d^f(t,t')\,dt'(\sigma_r^f(t,T+\delta) - \sigma_r^f(t,T)).\end{aligned}$$

The value of $\mu_r^f(t,t')$ with $t' \in [t, t+\delta)$ can be any of the number of posibilities that satisfy

$$\int_t^{t+\delta} \mu_r^f(t,t')\,dt' = g_r^f(t,t),$$

being

$$g_r^f(t,t) = -\int_t^{t+\delta} \sigma_r^f(t,t') \int_t^{t'} \sigma_r^f(t,t'')\,dt''\,dt' + \rho_{fr}^{fd}(t) \int_t^{t+\delta} \sigma_d^f(t,t')\,dt' \int_t^{t+\delta} \sigma_r^f(t,t')\,dt'.$$

In the case of the foreign spread curve, the same arguments give us that for $T \in [t+i\delta, t+(i+1)\delta)$, $i \geq 1$,

$$\mu_s^f(t,T) = \mu_s^f(t,T-i\delta) + \sum_{j=1}^{i} \frac{\partial g_s^f}{\partial T}(t,T-j\delta).$$

with

$$\begin{aligned}\frac{\partial g_s^f}{\partial T}(t,T) &= -(\sigma_s^f(t,T+\delta) - \sigma_s^f(t,T)) \int_T^{T+\delta} \sigma_s^f(t,t'')\,dt'' \\ &\quad + \rho_{fs}^{fd}(t)\sigma_d^f(t,T) \int_T^{T+\delta} \sigma_s^f(t,t')\,dt' + \rho_{fs}^{fd}(t) \int_t^{T} \sigma_d^f(t,t')\,dt'(\sigma_s^f(t,T+\delta) - \sigma_s^f(t,T)).\end{aligned}$$

The expression of $\mu_s^f(t,t')$ with $t' \in [t, t+\delta)$ is again any of the choices satisfying

$$\int_t^{t+\delta} \mu_s^f(t,t')\,dt' = g_s^f(t,t),$$

where

$$g_s^f(t,t) = -\int_t^{t+\delta} \sigma_s^f(t,t') \int_t^{t'} \sigma_s^f(t,t'')\,dt''\,dt'.$$

## 4 An example: dynamics in a Hull-White model

The Hull-White model proposes the following stochastic differential equation for the short rate:

$$dr(t) = (\theta(t) - a(t)r(t))\,dt + \sigma(t)dW(t).$$

We can solve the former equation to get:

$$r(t) = \mu(t) + b(t)\int_0^t f_s dW_s,$$

being

$$b(t) = e^{-\int_0^t a_s ds}, \qquad f(t) = \frac{\sigma(t)}{b(t)}.$$

We also define:

$$\beta(t) = \int_0^t b_s ds.$$

Recall that, in an HJM model,

$$r(t) = f(t,t) = f(0,t) + \int_0^t \mu(s,t)ds + \int_0^t \sigma(s,t)dW(s),$$

and therefore, we can then deduce that, in the Hull-White context:

$$\sigma(t,T) = b(t)f(t).$$

From this expression,

$$S(t,T) = \int_t^T \sigma(t,u)\,du = f(t)(\beta(T) - \beta(t)) \tag{4.1}$$

and

$$\int_0^t S(u,T)\,dW(u) = \beta(T)\int_0^t f(u)\,dW(u) - \int_0^t f(u)\beta(u)\,dW(u).$$

Observe that this is valid for any curve with a Hull-White dynamics. Also, the discussion above shows that the requirement given by equation (1.4) is fulfilled with a unique function $\mu_\gamma^\alpha(t,T)$ for each choice of $\gamma = r, s, d$ and $\alpha = f, d$. Once this is granted, from the point of view of the valuation with this model, the quantities whose dynamics need to be specified are bonds for the discount curves and Libor rates for the estimation and spread curves.

## 4.1 Domestic discounting bonds dynamics

By condition (3.1) and (4.1), the expressions for $A_d^d$ is

$$A_d^d(t,T) = \frac{1}{2}(\beta_d^d(T) - \beta_d^d(t))^2(f_d^d(t))^2,$$

and therefore

$$\int_0^t A_d^d(t',T)\,dt' = \frac{1}{2}(\beta_d^d(T))^2\int_0^t (f_d^d(s))^2\,ds - \beta_d^d(T)\int_0^t \beta_d^d(s)(f_d^d(s))^2\,ds + \frac{1}{2}\int_0^t (\beta_d^d(s))^2(f_d^d(s))^2\,ds$$

### 4.2 Domestic Libor dynamics

In terms of the reference curve, we have to express the dynamics of

$$L_r^d(t;T,U) = \left(\frac{B_r^d(t,T)}{B_r^d(t,U)} - 1\right)\frac{1}{m(T,U)},$$

where we write

$$\frac{B_r^d(t,T)}{B_r^d(t,U)} = \frac{B_r^d(0,T)}{B_r^d(0,U)} \exp\left\{\int_0^t (A_r^d(u,U) - A_r^d(u,T))\,du \int_0^t (S_r^d(u,U) - S_r^d(u,T))\,dW^{dr}(u)\right\}.$$

Condition (3.5) and (4.1) impose

$$\begin{aligned}
&A_r^d(u,U) - A_r^d(u,T)\\
&= -\frac{1}{2}(S_r^d(u,U) - S_r^d(u,T))^2 + \rho_{dr}^{dd}(u)S_d^d(u,U)(S_r^d(u,U) - S_r^d(u,T))\\
&= -\frac{1}{2}(f_r^d(u))^2(\beta_r^d(U) - \beta_r^d(T))^2 + \rho_{dr}^{dd}(u)f_r^d(u)f_d^d(u)(\beta_d^d(U) - \beta_d^d(u))(\beta_r^d(U) - \beta_r^d(T))
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_0^t (A_r^d(u,U) - A_r^d(u,T))\,du\\
&= -\frac{1}{2}(\beta_r^d(U) - \beta_r^d(T))^2 \int_0^t (f_r^d(u))^2\,du\\
&\quad +(\beta_r^d(U) - \beta_r^d(T))\left(\beta_d^d(U)\int_0^t \rho_{dr}^{dd}(u)f_r^d(u)f_d^d(u)\,du - \int_0^t \rho_{dr}^{dd}(u)f_r^d(u)f_d^d(u)\beta_d^d(u)\,du\right)
\end{aligned}$$

In terms of the spread curve, we have to express the dynamics of

$$L_r^d(t;T,U) = \left(\frac{B_d^d(t,T)B_s^d(t,T)}{B_d^d(t,U)B_s^d(t,U)} - 1\right)\frac{1}{m(T,U)},$$

where we write

$$\frac{B_s^d(t,T)}{B_s^d(t,U)} = \frac{B_s^d(0,T)}{B_s^d(0,U)} \exp\left\{\int_0^t (A_s^d(u,U) - A_s^d(u,T))\,du \int_0^t (S_s^d(u,U) - S_s^d(u,T))\,dW^{ds}(u)\right\}.$$

Condition (3.7) and (4.1), impose

$$\begin{aligned}
&A_s^d(u,U) - A_s^d(u,T)\\
&= -\frac{1}{2}(S_s^d(u,U) - S_s^d(u,T))^2 + \rho_{ds}^{dd}(u)S_d^d(u,T)(S_s^d(u,U) - S_s^d(u,T))\\
&= -\frac{1}{2}(f_s^d(u))^2(\beta_s^d(U) - \beta_s^d(T))^2 + \rho_{ds}^{dd}(u)f_s^d(u)f_d^d(u)(\beta_d^d(T) - \beta_d^d(u))(\beta_s^d(U) - \beta_s^d(T))
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_0^t (A_s^d(u,U) - A_s^d(u,T))\,du\\
&= -\frac{1}{2}(\beta_s^d(U) - \beta_s^d(T))^2 \int_0^t (f_s^d(u))^2\,du\\
&\quad +(\beta_s^d(U) - \beta_s^d(T))\left(\beta_d^d(T)\int_0^t \rho_{dr}^{dd}(u)f_s^d(u)f_d^d(u)\,du - \int_0^t \rho_{dr}^{dd}(u)f_s^d(u)f_d^d(u)\beta_d^d(u)\,du\right).
\end{aligned}$$

### 4.3 Foreign discounting bond dynamics

By condition (3.11) and (4.1), the expression for $A_d^f$ is

$$A_d^f(t,T) = \frac{1}{2}(f_d^f(t))^2(\beta_d^f(T) - \beta_d^f(t))^2 - \rho_S^{fd}(t)\sigma_S(t)f_d^f(t)(\beta_d^f(T) - \beta_d^f(t)).$$

Therefore

$$\begin{aligned}
&\int_0^t A_d^f(u,T)\,du \\
&= \frac{1}{2}(\beta_d^f(T))^2\int_0^t (f_d^f(u))^2\,du - \beta_d^f(T)\int_0^t (f_d^f(u))^2\beta_d^f(u)\,du + \frac{1}{2}\int_0^t (f_d^f(u))^2(\beta_d^f(u))^2\,du \\
&\quad -\beta_d^f(T)\int_0^t \rho_S^{fd}(u)\sigma_S(u)f_d^f(u)\,du + \int_0^t \rho_S^{fd}(u)\sigma_S(u)f_d^f(u)\beta_d^f(u)\,du.
\end{aligned}$$

### 4.4 Foreign Libor dynamics

Analogously to the domestic Libor dynamics, in terms of the reference curve, we have to express the dynamics of

$$L_r^f(t;T,U) = \left(\frac{B_r^f(t,T)}{B_r^f(t,U)} - 1\right)\frac{1}{m(T,U)},$$

where we write

$$\frac{B_r^f(t,T)}{B_r^f(t,U)} = \frac{B_r^f(0,T)}{B_r^f(0,U)}\exp\left\{\int_0^t (A_r^f(u,U) - A_r^f(u,T))\,du \int_0^t (S_r^f(u,U) - S_r^f(u,T))\,dW^{fr}(u)\right\}.$$

Condition (3.5) imposes

$$\begin{aligned}
&A_r^f(u,U) - A_r^f(u,T) \\
&= -\frac{1}{2}(S_r^f(u,U) - S_r^f(u,T))^2 + \rho_{fr}^{fd}(u)S_d^f(u,U)(S_r^f(u,U) - S_r^f(u,T)) \\
&= -\frac{1}{2}(f_r^f(u))^2(\beta_r^f(U) - \beta_r^f(T))^2 + \rho_{fr}^{fd}(u)f_r^f(u)f_d^f(u)(\beta_d^f(U) - \beta_d^f(u))(\beta_r^f(U) - \beta_r^f(T))
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_0^t (A_r^f(u,U) - A_r^f(u,T))\,du \\
&= -\frac{1}{2}(\beta_r^f(U) - \beta_r^f(T))^2\int_0^t (f_r^f(u))^2\,du \\
&\quad +(\beta_r^f(U) - \beta_r^f(T))\left(\beta_d^f(U)\int_0^t \rho_{fr}^{fd}(u)f_r^f(u)f_d^f(u)\,du - \int_0^t \rho_{fr}^{fd}(u)f_r^f(u)f_d^f(u)\beta_d^f(u)\,du\right)
\end{aligned}$$

In terms of the spread curve, we have to express the dynamics of

$$L_r^f(t;T,U) = \left(\frac{B_d^f(t,T)B_s^f(t,T)}{B_d^f(t,U)B_s^f(t,U)} - 1\right)\frac{1}{m(T,U)},$$

where we write

$$\frac{B_s^f(t,T)}{B_s^f(t,U)} = \frac{B_s^f(0,T)}{B_s^f(0,U)} \exp\left\{ \int_0^t (A_s^f(u,U) - A_s^f(u,T))\, du \int_0^t (S_s^f(u,U) - S_s^f(u,T))\, dW^{ds}(u) \right\}.$$

Condition (3.7) imposes

$$\begin{aligned} & A_s^f(u,U) - A_s^f(u,T) \\ &= -\frac{1}{2}(S_s^f(u,U) - S_s^f(u,T))^2 + \rho_{fs}^{fd}(u) S_d^f(u,T)(S_s^f(u,U) - S_s^f(u,T)) \\ &= -\frac{1}{2}(f_s^f(u))^2(\beta_s^f(U) - \beta_s^f(T))^2 + \rho_{fs}^{fd}(u) f_s^f(u) f_d^f(u)(\beta_d^f(T) - \beta_d^f(u))(\beta_s^f(U) - \beta_s^f(T)) \end{aligned}$$

Therefore

$$\begin{aligned} & \int_0^t (A_s^f(u,U) - A_s^f(u,T))\, du \\ &= -\frac{1}{2}(\beta_s^f(U) - \beta_s^f(T))^2 \int_0^t (f_s^f(u))^2\, du \\ & \quad +(\beta_s^f(U) - \beta_s^f(T)) \left( \beta_d^f(T) \int_0^t \rho_{fr}^{fd}(u) f_s^f(u) f_d^f(u)\, du - \int_0^t \rho_{fr}^{fd}(u) f_s^f(u) f_d^f(u) \beta_d^f(u)\, du \right). \end{aligned}$$

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