Lévy-VaR and the Protection of Banks and Insurance Companies

A joint work by

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Testing for Jumps

Aït-Sahalia and Jacod
(2009, Annals of Statistics)
constructed the estimator :

$$\widehat{S}(p,k,\Delta n)_t = \frac{\sum_{i=1}^{\lfloor t/(k\Delta_n)\rfloor} |X_{ik\Delta_n} - X_{(i-1)k\Delta_n}|^p}{\sum_{i=1}^{\lfloor t/\Delta_n\rfloor} |X_{i\Delta_n} - X_{(i-1)\Delta_n}|^p}$$

Testing for Jumps

It can be proved that :

$$\widehat{S}(4, 2, \Delta n)_t \rightarrow 1$$

for a process that displays jumps, and that :

$$\widehat{S}(4, 2, \Delta n)_t \rightarrow 2$$

for a purely continuous process.

Testing for Jumps

Day	Stat	Cont. Thresh.	Disc. Thresh.
26/01/09	1,03	0,88	1,47
27/01/09	1,06	1,89	1,23
28/01/09	0,74	1,23	1,47
29/01/09	0,69	1,60	1,35
30/01/09	0,96	1,69	1,32

Case Study : Dynamics of Total

A History of Finance
From Brownian Motion to Lévy Proc.

Bachelier (1900)

Mandelbrot (1963)

Samuelson (1965)

Merton (1976)

Carr, Geman, Madan and Yor (2002)

A History of Insurance From Poisson to Lévy Proc.

Lundberg (1903)

Cramer (1930)

Gerber (1970)

Dufresne, Gerber and Shiu (1991)

Klüppelberg, Kyprianou, Maller (2003)

A New Financial Framework
Lévy Processes

Continuous Time Random Walk

Stationary and Independent Increments

Based on Infinitely Divisible Laws :

$X = X_1 + ... + X_i + ... + X_n$

Spectrum Satisfies

$\Phi(t) = E\left(e^{iuX_t}\right) = \Phi(1)^t$

A New Financial Framework
Lévy-Khintchine Formula

The characteristic function
of any marginal X_t satisfies :

$$E(e^{iuX_t}) = e^{iatu - \frac{1}{2}btu^2 - t\int_{\mathbb{R}}\left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|\leq 1\}}\right)\nu(dx)}$$

where :

a is a drift term,

b is a diffusion term,

and $\nu(.)$ is a Lévy measure.

What we do *not* Assume
Stable Distributions

Stability means

$$AX_1 + BX_2 = CX + D$$

Or :

$$X_1 + ... + X_i + ... + X_n = C_n X + D_n$$

where X admits the same law as all X_i.

Same behavior at all scales : Fractality

Asymptotic Patterns

Gaussian law is :

$$\# \ e^{-x^2}$$

Paretian-like behavior means :

$$\# \ \frac{C}{x^\alpha}$$

Desired non-stable Lévy tails :

$$\# \ C\frac{e^{-\beta x}}{x^\alpha}$$

(Mixed Exponential and Power Functions
= Semi-Violent Extreme Risks)

A New Paradigm
Asymmetric Laplace Motion

A fundamental tool : the Gamma Process.

This is a Lévy process which jumps only upwards.

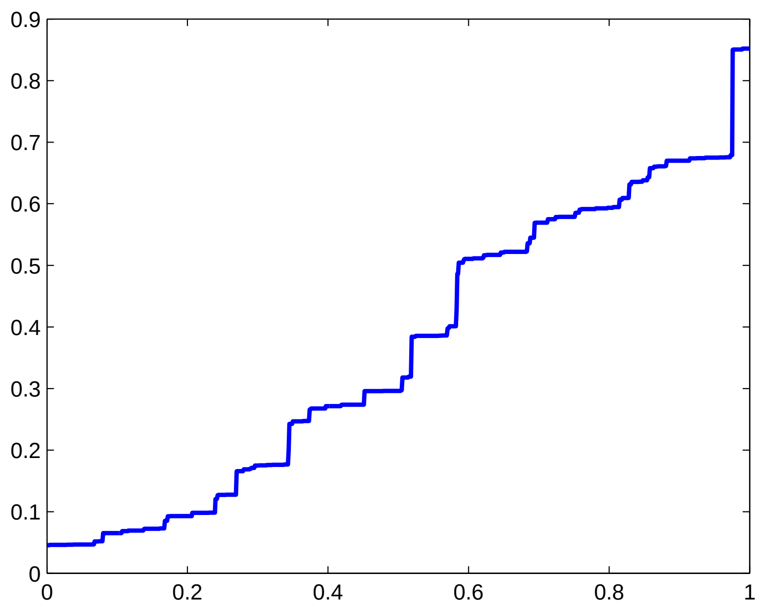
Marginal Density :

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

Spectrum :

$$E\left[e^{iuX}\right] = \left(1 - \frac{iu}{\lambda}\right)^{-\alpha}$$

Simulation of a Gamma Process



A New Paradigm
Asymmetric Laplace Motion

A first construction of
the Variance Gamma Process :

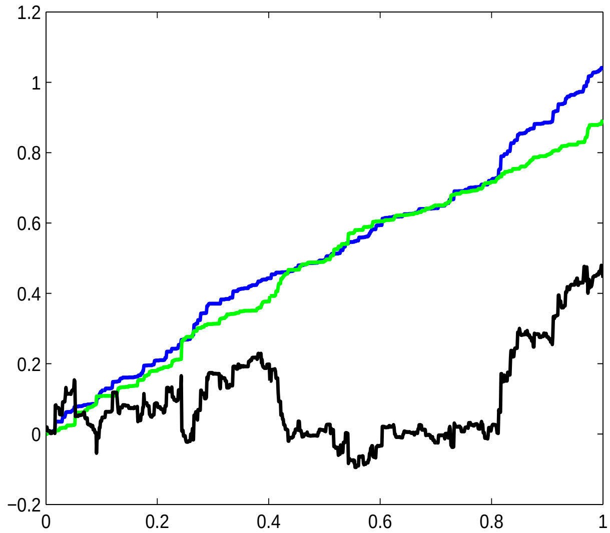
$$VG_t = G_t^1 - G_t^2$$

where G^1 accounts for positive jumps,
and G^2 for negative jumps.

G^1 and G^2 are reduced Gamma processes :

$$\alpha = \lambda = \frac{1}{\nu}$$

Simulation of a VG Process as a Difference of 2 Gamma Proc.



VG (black) = G (blue) - G(green)

A New Paradigm
Asymmetric Laplace Motion

A second construction of
the Variance Gamma Process :

$$VG_t = \mu G_t + \sigma z(G_t)$$

where :

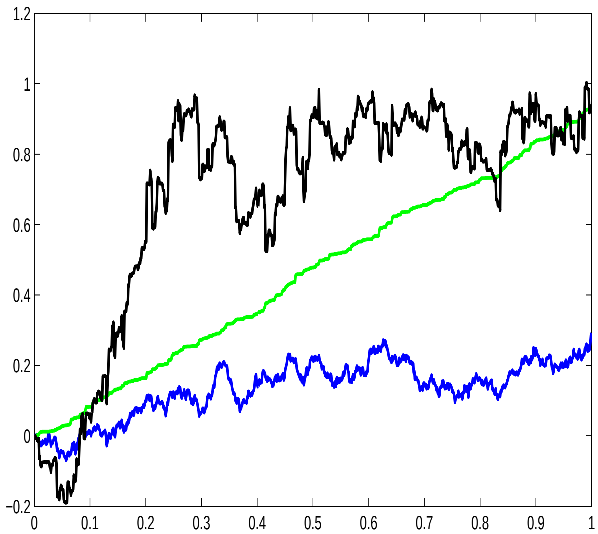
$$ABM_t = \mu t + \sigma z_t$$

is an arithmetic Brownian motion,
and G_t is a Gamma time change.

Equivalently :

$$VG_t = ABM(G_t)$$

Simulation of a VG Process as a Time-Changed ABM



VG (black) = ABM (blue)
taken at time G(green)

A New Paradigm
Asymmetric Laplace Motion

Finally, asymmetric Laplace motion is :

$$ALaM_t = \delta t + V G_t$$

Or, equivalently :

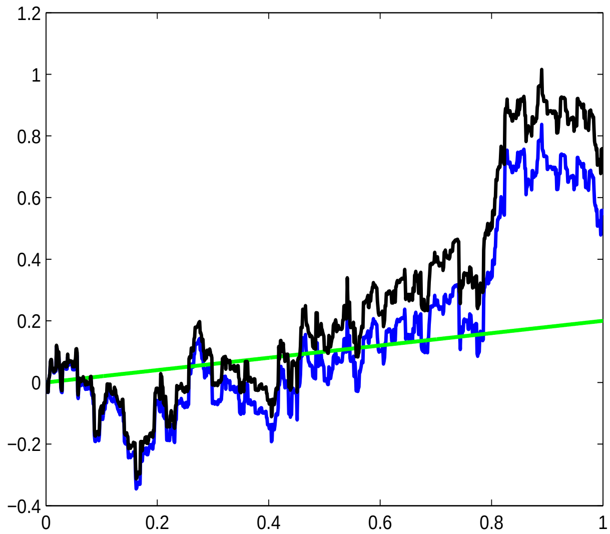
$$ALaM_t = \delta t + \mu G_t + \sigma z(G_t)$$

Well-designed process :
Four parameters <-> Four Moments

The spectrum is :

$$E\left[e^{iuX_1}\right] = \frac{e^{iu\delta}}{\left(1 + \frac{\sigma^2 \nu u^2}{2} - i\mu\nu u\right)^{\frac{1}{\nu}}}$$

Simulation of an ALaM



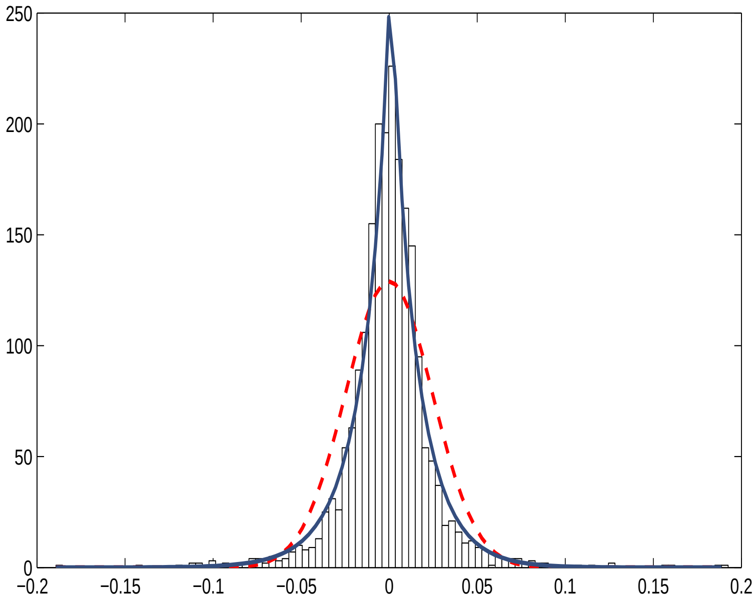
ALaM (black) = VG (blue) + Drift (green)

Typical ALaM Parameter Values

Quote	μ	σ	ν	δ
AXA	-0,0006	0,0290	1,1938	0
BNP	-0,0011	0,0229	1,1519	0,0010
Carrefour	0,0006	0,0189	0,9549	-0,0010
PPR	0	0,0235	1,3933	-0,0006
Total	-0,0013	0,0173	0,7529	0,0013
Vinci	0,0003	0,0191	1,1516	0
CAC40	-0,0011	0,0154	0,9603	0,0008

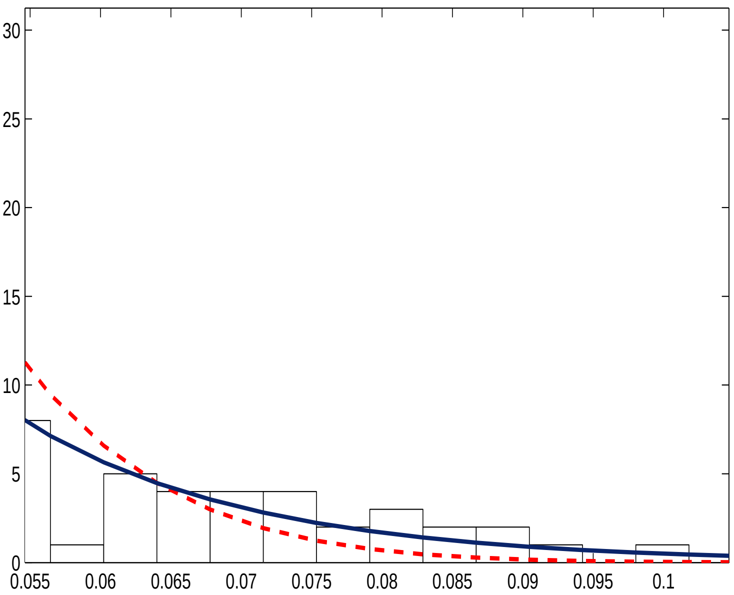
Calibration is done over the period
2001 - 15th of April 2009
(Maximum Likelihood)

Fitting Empirical Distributions



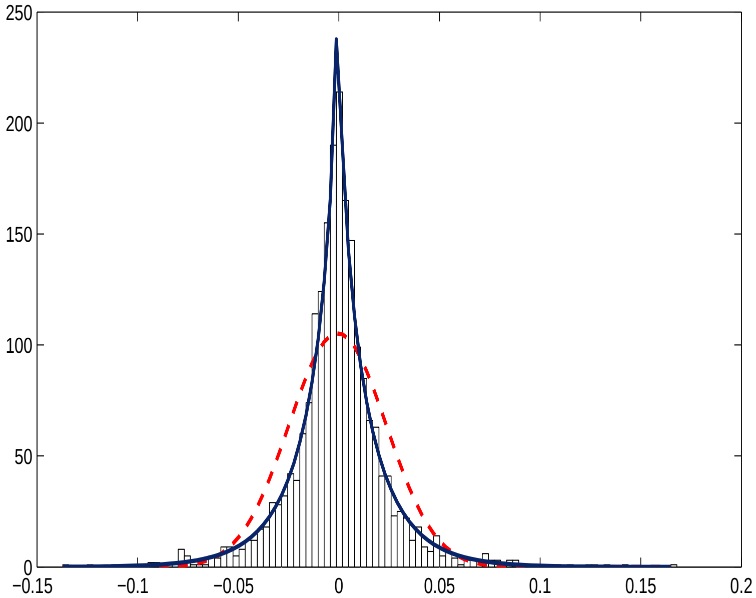
Distribution Fit with BNP daily quotes between 2001 and 15th of April 2009

Fitting Empirical Distributions



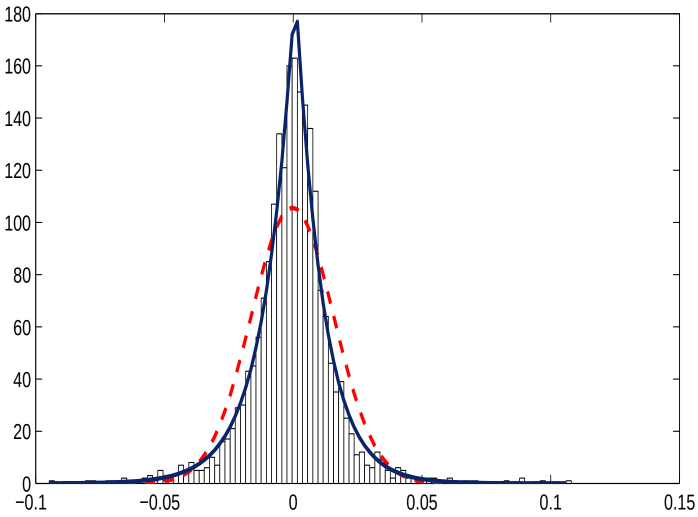
Tail Fit with BNP daily quotes between 2001 and 15th of April 2009

Fitting Empirical Distributions



Distribution Fit with PPR daily quotes between 2001 and 15th of April 2009

Fitting Empirical Distributions



Distribution Fit with CAC 40 daily quotes between 2001 and 15th of April 2009

A Comment
Moments of AL Motions

$$E(X_1) = \mu + \delta$$

$$\left(\quad m^n(X_1) = E\left([X_1 - E(X_1)]^n\right) \quad \right)$$

$$m^2(X_1) = \text{Variance}(X_1) = \sigma^2 + \mu^2\nu$$

$$m^3(X_1) = 2\mu^3\nu^2 + 3\mu\sigma^2\nu$$

$$m^4(X_1) = 6\mu^4\nu^3 + 12\mu^2\sigma^2\nu^2 + 3\sigma^4\nu + 3\mu^4\nu^2 + 6\mu^2\sigma^2\nu + 3\sigma^4$$

Computing Value-at-Risk

Value-at-Risk is defined by :

$$F(\text{VaR}_\beta) = 1 - \beta$$

where β is the no-ruin probability,
and F is the c.d.f. of the variable under study.

Goal of our paper :

Easy Computation of VaR
with Asymmetric Laplace Motions

Computing Value-at-Risk

A first algorithm :

In general, the c.d.f. F and the density f
are unknown,
but the characteristic function ϕ is known.

The first algorithm reads :
$\phi \longrightarrow$ Inverse Fourier Transform $\longrightarrow f$
$f \longrightarrow$ Quadrature $\longrightarrow F$
Perform root search on this F to obtain VaR

Computing Value-at-Risk

<u>A second algorithm :</u>

In fact, one has the following formula :

$$F(x) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \frac{\phi(t)}{it} dt$$

The second algorithm then reads :
$\phi \longrightarrow$ Inverse Fourier Transform $\longrightarrow F$
Perform root search on this F to obtain VaR

Problem : Stability around $t = 0$!

Computing Value-at-Risk

Our algorithm :

We obtained the formula :

$$F(x) = \frac{e^{\alpha x}}{2\pi} \int_{-\infty}^{+\infty} e^{ixv} \, \frac{\phi(-v + i\alpha)}{\alpha + iv} \, dv$$

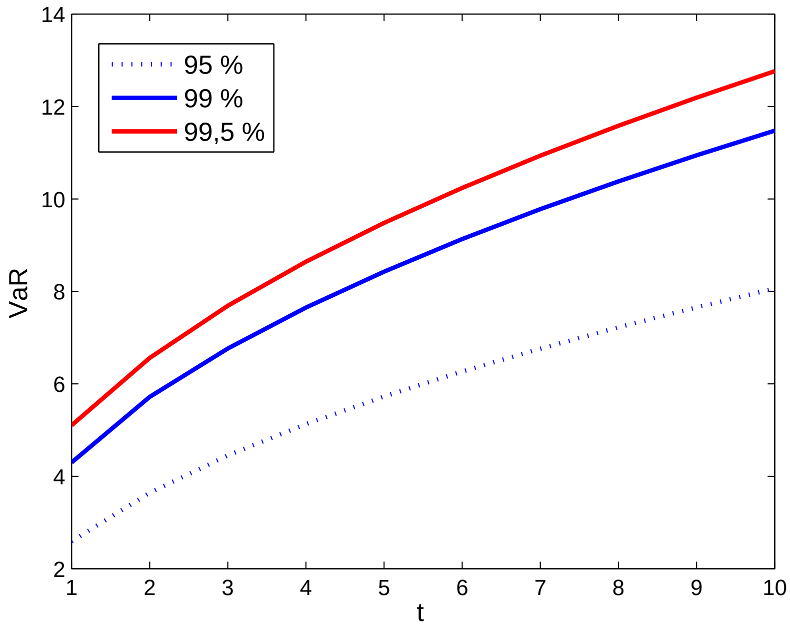
for any positive real number α.

The general approach remains identical :

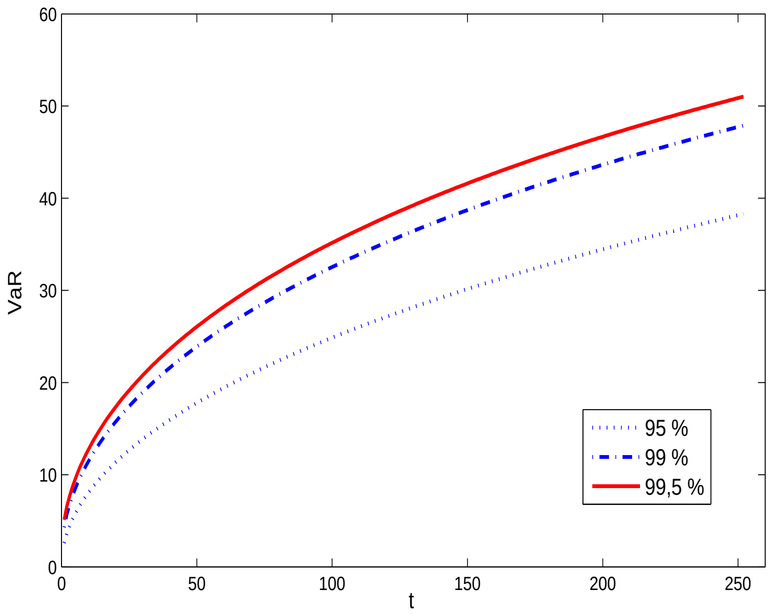
$\phi \longrightarrow$ Inverse Fourier Transform $\longrightarrow F$

Perform root search on this F to obtain VaR

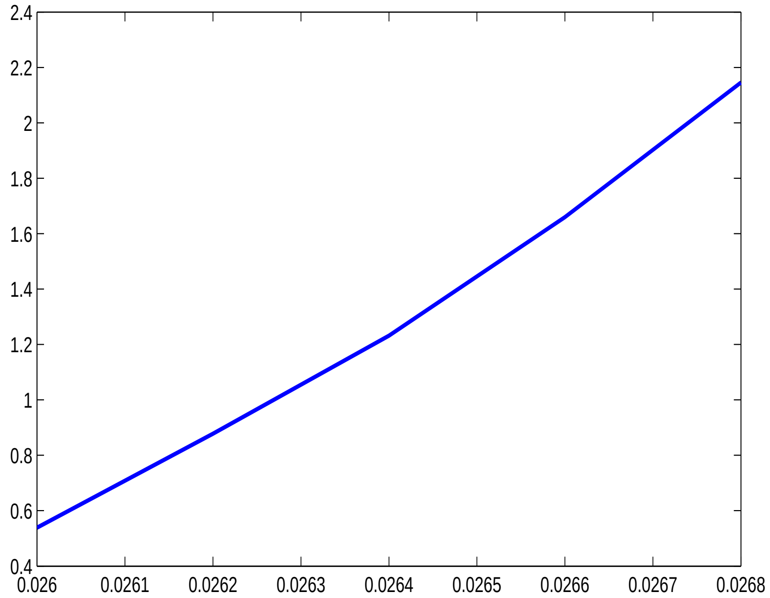
Short-Term VaR



Medium-Term VaR



Implicit Process



Gives the implicit ν
for a given value of VAR

(All other Parameters are those of CAC 40)

Next Step

Basle Agreements Incite Banks
to Post a VaR equal to
3 times the Gaussian VaR

To which ALaM corresponds this
assumption ?

Put Differently what Implicit Model do
Regulators Have in Mind ?