An "Almost Exact" Simulation Method for the Heston Model

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Madrid, October 18, 2006

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The Heston Model

$$dF_t = \sqrt{V_t} F_t dW_1$$

$$dV_t = \underbrace{\kappa_t (\theta_t - V_t) dt}_{\text{mean reverting term}} + \sigma_t \sqrt{V_t} dW_2$$

$$dW_1 dW_2 = \rho_t dt$$

- V_t: Variance (not volatility)
- σ_t: Volatility of variance
- θ_t: Long term variance
- κ_t: Mean reversion rate
- ρ_t: Correlation between changes in variance and forward

- **In Fourier Space, two dimensional Heston PDE becomes one dimensional**

=> Vanilla prices have closed form solutions

The Heston Model (II)

- Pricing involves numerically integrating analytical solutions of Riccati Equations.

- Time dependent case (with piecewise constant parameters) is no more complicated than the flat case!!!

- The Riccati Equations must be solved in sections of time in which the parameters are constant, working backwards from maturity.

- Computational time ≈ number of time sections * time to calculate flat Heston.

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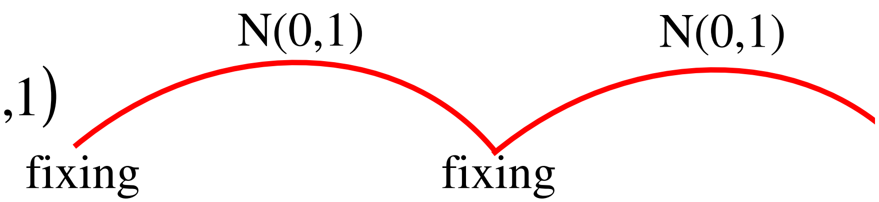
An Exact Simulation Method

- **Black Scholes has analytical solution for SDE**
  => we can "long jump" in Monte Carlo

$$S_t = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}N(0,1)}$$



- **In Stochastic Vol Models we have to do many little timesteps (SLOW!)**

- **Much research on better discretisation methods than Euler (e.g. Predictor-Corrector, Milstein…)**

- **Broadie and Kaya presented an "Exact Simulation Method" in their paper of 2003**

An Exact Simulation Method (II)

Advantages of an exact simulation method are

- **We need only simulate the dates of importance for the financial product we are modeling.**
- **The simulations have no bias in the error so we can construct valid Confidence Intervals**
- **The Order of Convergence improves from ⅓ to ½**
- **Greeks can be calculated using the same "tricks" as with Black-Scholes model because the forward price is lognormal conditional upon the path of the variance**

An Exact Simulation Method (III)

$$F_t = F_u \exp\left[-\frac{1}{2}\int_u^t V_s ds + \rho \int_u^t \sqrt{V_s} dW_s^{(1)} + \sqrt{1-\rho^2} \int_u^t \sqrt{V_s} dW_s^{(2)} \right]$$

$$V_t = V_u + \kappa\theta(t-u) - \kappa \int_u^t V_s ds + \sigma \int_u^t \sqrt{V_s} dW_s^{(1)}$$

$$dW_s^{(1)} dW_s^{(2)} = 0$$

- **Generate a sample from the distribution of V_t given V_u**
- **Generate a sample from the distribution of $\int_u^t V_s ds$ given V_t and V_u**
- **Recover $\int_u^t \sqrt{V_s} dW_s^{(1)}$ from the above equation given V_t, V_u and $\int_u^t V_s ds$**
- **Generate a sample from the distribution of F_t given $\int_u^t \sqrt{V_s} dW_s^{(1)}$ and $\int_u^t V_s ds$**

An Exact Simulation Method (IV)

1) Generate a sample from the distribution of V_t given V_u

- The distribution is known and is a non-central chi-squared distribution
- Samples can be generated using combinations of Poisson, Gamma and Normal variates
- Sankaran (1963) shows that one can transform a non-central chi squared variate into an approximately normal variate. We use this to calculate confidence intervals for V_t which we use in the next section.

An Exact Simulation Method (V)

2) Generate a sample from the distribution of $\int_u^t V_s ds$ given V_t and V_u

- Brute force method: the CDF is given by a numerical integral of the known characteristic function of the distribution. We can generate samples from uniform deviates by inverting the CDF using Newton's method.
- The characteristic function is computationally expensive as it involves modified Bessel functions of the first kind with a complex argument
- We should think about caching as much as we can!

An Exact Simulation Method (VI)

The characteristic function is given by

$$\Phi(a;V_u,V_t)=\frac{\gamma(a)e^{-\frac{1}{2}(\gamma(a)-\kappa)(t-u)}\left(1-e^{-\kappa(t-u)}\right)}{\kappa\left(1-e^{-\gamma(a)(t-u)}\right)}$$

$$\times\exp\left\{\frac{V_u+V_t}{\sigma^2}\left[\frac{\kappa\left(1+e^{-\kappa(t-u)}\right)}{1-e^{-\kappa(t-u)}}-\frac{\gamma(a)\left(1+e^{-\gamma(a)(t-u)}\right)}{1-e^{-\gamma(a)(t-u)}}\right]\right\}$$

$$\times\frac{I_{0.5d-1}\left[\sqrt{V_uV_t}\,\frac{4\gamma(a)e^{-0.5\gamma(a)(t-u)}}{\sigma^2\left(1-e^{-\gamma(a)(t-u)}\right)}\right]}{I_{0.5d-1}\left[\sqrt{V_uV_t}\,\frac{4\kappa e^{-0.5\kappa(t-u)}}{\sigma^2\left(1-e^{-\kappa(t-u)}\right)}\right]}$$

Bessel functions of the first kind with complex argument

An Exact Simulation Method (VII)

3) Recover $\int_u^t \sqrt{V_s}\, dW_s^{(1)}$ **from the above equation given**

V_t, V_u **and** $\int_u^t V_s\, ds$

$$\int_u^t \sqrt{V_s}\, dW_s^{(1)} = \left(\frac{1}{\sigma}\right)\left(V_t - V_u - \kappa\theta(t-u) + \kappa\int_u^t V_s\, ds\right)$$

An Exact Simulation Method (VIII)

4) Generate a sample from the distribution of F_t given

$\int_u^t \sqrt{V_s}\, dW_s^{(1)}$ and $\int_u^t V_s\, ds$

- Conditional on $\int_u^t \sqrt{V_s}\, dW_s^{(1)}$ and $\int_u^t V_s\, ds$ F_t is lognormal

  with drift rate

$$\hat{\mu} = -\frac{1}{2t}\int_u^t V_s\, ds + \frac{\rho}{t}\int_u^t \sqrt{V_s}\, dW_s^{(1)}$$

  and volatility

$$\hat{\sigma} = \sqrt{\frac{1-\rho^2}{t}}\sqrt{\int_u^t V_s\, ds}$$

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An *Almost* Exact Simulation Method

- **The Exact Simulation Method is only appropriate for options which depend on only one future point in time because the characteristic function otherwise depends on three variables, a, V_t and V_u. We cannot efficiently cache the characteristic function in three dimensions.**

- **If we look closely at the characteristic function we notice that it depends on V_t and V_u via their arithmetic and geometric mean which, at least in expectation, are approximately equal…**

An Almost Exact Simulation Method (II)

The characteristic function is given by

$$\Phi(a;V_u,V_t) = \frac{\gamma(a)e^{-\frac{1}{2}(\gamma(a)-\kappa)(t-u)}\left(1-e^{-\kappa(t-u)}\right)}{\kappa\left(1-e^{-\gamma(a)(t-u)}\right)}$$

$$\times\exp\left\{\frac{2}{\sigma^2}\frac{V_u+V_t}{2}\left[\frac{\kappa\left(1+e^{-\kappa(t-u)}\right)}{1-e^{-\kappa(t-u)}}-\frac{\gamma(a)\left(1+e^{-\gamma(a)(t-u)}\right)}{1-e^{-\gamma(a)(t-u)}}\right]\right\}$$

$$\times\frac{I_{0.5d-1}\left[\sqrt{V_uV_t}\frac{4\gamma(a)e^{-0.5\gamma(a)(t-u)}}{\sigma^2\left(1-e^{-\gamma(a)(t-u)}\right)}\right]}{I_{0.5d-1}\left[\sqrt{V_uV_t}\frac{4\kappa e^{-0.5\kappa(t-u)}}{\sigma^2\left(1-e^{-\kappa(t-u)}\right)}\right]}$$

Arithmetic mean

Geometric mean

An Almost Exact Simulation Method (III)

- The idea is to replace both the geometric mean and the arithmetic mean by a weighted average of the two

$$z = \omega \frac{1}{2}\left(V_u + V_t\right) + \left(1 - \omega\right)\sqrt{V_u V_t}$$

- If the method is a good one we should find that the results are virtually independent of ω.

- In fact, observed differences for values of ω between 0 and 1 were in the order of a fraction of a basis point for options up to 5 years maturity.

An Almost Exact Simulation Method (IV)

The characteristic function is now given by

$$\Phi(a;z) = \frac{\gamma(a)e^{-\frac{1}{2}(\gamma(a)-\kappa)(t-u)}\left(1-e^{-\kappa(t-u)}\right)}{\kappa\left(1-e^{-\gamma(a)(t-u)}\right)}$$

$$\times\exp\left\{\frac{2}{\sigma^2}z\left[\frac{\kappa\left(1+e^{-\kappa(t-u)}\right)}{1-e^{-\kappa(t-u)}}-\frac{\gamma(a)\left(1+e^{-\gamma(a)(t-u)}\right)}{1-e^{-\gamma(a)(t-u)}}\right]\right\}$$

$$\times\frac{I_{0.5d-1}\left[z\frac{4\gamma(a)e^{-0.5\gamma(a)(t-u)}}{\sigma^2\left(1-e^{-\gamma(a)(t-u)}\right)}\right]}{I_{0.5d-1}\left[z\frac{4\kappa e^{-0.5\kappa(t-u)}}{\sigma^2\left(1-e^{-\kappa(t-u)}\right)}\right]}$$

and can be efficiently cached in two dimensions.

An Almost Exact Simulation Method (V)

- We have managed to take the heavy work out of the Monte Carlo loop. Now the overhead does not depend on the number of simulations.

- Most importantly, we can now apply the method to options which depend on several points in time (very common in Equity).

- We can improve our approximation by using knowledge of the first two moments of the true distribution. (Requires only a few evaluations of the characteristic function.)

- We use the true and approximate means and variances to shift and scale our approximate distribution. In this way we ensure to match the first two moments of the true distribution.

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Conclusions and Extensions

- We can calibrate the time dependent Heston model as efficiently as the flat Heston model.

- The Almost Exact Simulation method outperforms other published discretisation methods. It has virtually no bias. Techniques for simulating the Greeks can be used unchanged.

- It is relatively straightforward to apply the Almost Exact Simulation method to some other well known jump processes.

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