Pricing Credit Derivatives and Measuring Credit Risk in Multifactor Models

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Objective

• Develop fast, accurate approximations for
  – measuring credit risk
  – pricing credit derivatives
in multifactor Gaussian copula models

• Why multifactor Gaussian copula?
  – Gaussian copula is the industry standard
  – Skew in implied correlation contradicts single factor
  – For risk management, number of factors should be
    close to number of factors in an equity model

• Obstacles are calibration and computation
Risk Measurement and Pricing

• Risk measurement problem:
  – Find the distribution of losses in a credit portfolio over a fixed horizon; calculate risk measure(s)

• Pricing problem
  – Price a CDO tranche
  – Price depends on the distribution of losses in the underlying collateral at each coupon date

In both cases, modeling the distribution of losses is key
Risk Measurement and Pricing

- Risk measurement problem:
  - Find the distribution of losses in a credit portfolio over a fixed horizon; calculate risk measure(s)
Risk Measurement and Pricing

• CDO Pricing problem:
  – Find the distribution of losses in a credit portfolio over a fixed horizon; calculate option-like payoff

For both problems, modeling the loss distribution is key
Outline

• Gaussian copula model of default times
• Multifactor models – advantages and difficulties
• Approximations
  – Correlation expansion: use a Taylor expansion around a zero-factor or single-factor model
  – Quadratic transform approximation: approximate the characteristic function of a multifactor model and invert the approximation
Modeling Default Times: Gaussian Copula
Modeling Default Times

In order to
• measure portfolio credit risk
• price portfolio credit derivatives
need to model the joint distribution of the default times
\((\tau_1, \ldots, \tau_m)\) of the assets in the portfolio

Copula perspective:

i. specify the marginal distribution for each asset
   – use information from credit spreads, ratings

ii. specify the dependence ("correlation") between default times – this is the copula
Default Times: Marginal Distributions

\( \tau_k = \) time to default for name \( k \) in underlying portfolio

- Default time \( \tau_k \) determined by *hazard rate function* for that name, \( h_k(t) \)

- This is, e.g., the CDS spread for that name

- Distribution of time to default for name \( k \):

\[
P(\tau_k \leq t) = 1 - \exp \left( - \int_0^t h_k(s) \, ds \right)
\]
Simulating Default Time

- Generate Uniform\([0,1]\) \(U\)
- Apply inverse distribution to produce default time \(\tau_k\)
Gaussian Copula: Joint Distribution

$\Phi$ = normal dist.

$\Phi(X_1)$ $\rightarrow$ $U_1$

$\Phi(X_2)$ $\rightarrow$ $U_2$

Unif[0,1]

Correlations in latent variables determine dependence between default times.
Dependent Defaults: Gaussian Copula

\[
\begin{align*}
X_1 & \to U_1 \to \tau_1 \\
X_2 & \to U_2 \to \tau_2 \\
& \quad \vdots \\
X_N & \to U_N \to \tau_N
\end{align*}
\]

Dependence between default times \(\tau_1, \tau_2, \ldots, \tau_N\) determined by correlation matrix \(\Sigma\)
Losses at a Fixed Horizon

Consider losses at a fixed horizon $T$

$k$th asset defaults over this horizon if $\tau_k \leq T$

Equivalently, if its latent variable $X_k$ falls below some threshold $v_k$

$p_k =$ marginal default probability
Loss Distribution at a Fixed Horizon

Loss given default $c_k$ for $k$th asset

Portfolio loss at a fixed horizon is

$$L = \sum_{k=1}^{m} c_k \mathbb{1}\{X_k \leq v_k\}$$
Once We Have the Portfolio Loss $L$...

...we can calculate VaR or some other portfolio risk measure

... we can price a CDO
Factor Models and Multifactor Models
Factor Models

- Recall the latent variables \((X_1, \ldots, X_N)\) have a multivariate normal distribution
- Factor representation

\[
X_k = a_{k1} Z_1 + \cdots + a_{kd} Z_d + b_k \epsilon_k
\]

- \(Z_1, \ldots, Z_d, \epsilon_k\) independent standard normals
Factor Models

- Latent variables loosely related to firm value through Merton model
- Factors may have economic interpretation

<table>
<thead>
<tr>
<th>Firm Value</th>
<th>Market</th>
<th>Energy</th>
<th>Auto</th>
<th>Telecomm</th>
<th>Europe</th>
<th>Japan</th>
<th>Specific Risk</th>
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<tbody>
<tr>
<td>Name 1</td>
<td>$X_1$</td>
<td>$a_{11}$</td>
<td>$a_{12}$</td>
<td></td>
<td>$a_{1d}$</td>
<td></td>
<td>$b_1$</td>
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<tr>
<td>Name 2</td>
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<td>$a_{21}$</td>
<td></td>
<td>$a_{24}$</td>
<td>$a_{2,d-1}$</td>
<td></td>
<td>$b_2$</td>
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<tr>
<td>Name 3</td>
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<td>$b_3$</td>
</tr>
<tr>
<td>Name m</td>
<td>$X_m$</td>
<td>$a_{m1}$</td>
<td>$a_{m3}$</td>
<td></td>
<td>$a_{md}$</td>
<td></td>
<td>$b_m$</td>
</tr>
</tbody>
</table>
Factor Models

\[ X_k = a_{k1}Z_1 + \cdots + a_{kd}Z_d + b_k \varepsilon_k \]

\[ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{dd} \end{bmatrix} \]

Correlation matrix
\[ AA' + \text{diag}(b_1^2, \ldots, b_m^2) \]

\[ \rho_{ij} = \sum_{k=1}^{m} a_{ik} a_{kj} \]

Defaults become independent conditional on factors
Single-Factor Gaussian Copula

• Factor representation

\[ X_k = \sqrt{\rho} Z + \sqrt{1 - \rho} \varepsilon_k \]

• Correlation matrix of \((X_1, \ldots, X_N)\)

\[
\begin{pmatrix}
1 & \rho & \cdots & \rho \\
\rho & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \rho \\
\rho & \cdots & \rho & 1
\end{pmatrix}
\]
Implied Correlation Skew

Single-factor Gaussian copula can’t match market data:
Different tranches priced at different correlations
Multifactor Models

- There are many ways of generating a skew
  - E.g., change the copula
- What about multiple factors?
- What kind of factor structure would generate the typical pattern of implied correlation?
- Need more correlation in rare defaults of highest quality credits
Example

200 names in four blocks of 50 each

LGD = 1

\[
\begin{bmatrix}
1\% \\
\vdots \\
3\% \\
5\% \\
7\%
\end{bmatrix}
\begin{bmatrix}
.3 & .4 & .5 & .6 \\
\vdots & \vdots & \vdots & \vdots \\
3 & 4 & 5 \\
\vdots & \vdots \\
3 & 4
\end{bmatrix}
\]

default prob.s = \vdots, factor loadings = \vdots

50 highest-rated names have highest correlation
Implied Correlation: 4-Factor Model

![Graph showing the relationship between attachment point and implied correlation. The graph plots correlation on the y-axis and attachment point on the x-axis. The curve indicates an increasing correlation with increasing attachment point.]
Gaussian Copula Calculations

• “zero-factor” model (independent defaults) is tractable

• For a single-factor model:
  – Condition on the factor to make defaults independent
  – Integrate over the single factor

• Multifactor model?
  – Condition on factors to make defaults independent
  – Dimension of integration equals number of factors
  – When the number of factors is large, this requires Monte Carlo

• As an alternative, we develop approximations
Correlation Expansions
Correlation Expansions

- Given multifactor correlation matrix with entries $\rho_{ij}$
- Introduce family of correlation matrices with off-diagonal entries $t\rho_{ij}$, $0 \leq t \leq 1$.

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & t\rho_{12} & \cdots & t\rho_{1d} \\
t\rho_{21} & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & t\rho_{d-1,d} \\
t\rho_{d1} & \cdots & t\rho_{d,d-1} & 1
\end{pmatrix}
\begin{pmatrix}
1 & \rho_{12} & \cdots & \rho_{1d} \\
\rho_{21} & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \rho_{d-1,d} \\
\rho_{d1} & \cdots & \rho_{d,d-1} & 1
\end{pmatrix}
\]

independent model

\begin{align*}
t = 0 \\
t = 1
\end{align*}

target model

- Approximate target by expanding in $t$
Correlation Expansions

• Let $L$ = portfolio loss at a coupon date
• $E_t$ = expectation using correlations $t \rho_{ij}$
• Tranche price reduces to calculating

$$E_t (L - y)^+, \text{ at } t = 1$$

• Expansion

$$E_t (L - y)^+ = \delta_0 + \delta_1 t + \frac{1}{2} \delta_2 t^2 + \cdots$$

Coefficients $\delta_i$ can be calculated in independent-obligor model
Correlation Expansions

Kibble (1945): For multivariate normal $X_1, \ldots, X_m$ with correlations $t \rho_{ij}$

$$P_t(X_1 > x_1, \ldots, X_m > x_m) = b_0 + b_1 t + \frac{1}{2} b_2 t^2 + \cdots$$

$$b_n = \sum_{1 \leq j_1 < k_1 \leq m} \rho_{j_1 k_1} \cdots \sum_{1 \leq j_n < k_n \leq m} \rho_{j_n k_n} B_{#1}(x_1) \cdots B_{#m}(x_m)$$

$$B_n(x) = \phi(x) H_{n-1}(x), \quad H_k \text{ Hermite polynomials}$$

We develop efficient method to calculate coefficients as weighted sum of independent probabilities
Approximating the Loss Distribution

Example: Suppose LGDs are 1, 2, 3, and 4

\[ P(L \geq 10) = P(X_1 > \nu_1, X_2 > \nu_2, X_3 > \nu_3, X_4 > \nu_4) \]

\[ P(L \geq 9) = P(X_2 > \nu_2, X_3 > \nu_3, X_4 > \nu_4) \]

\[ P(L \geq 7) = P(X_1 > \nu_1, X_2 > \nu_2, X_3 \leq \nu_3, X_4 > \nu_4) + P(X_3 > \nu_3, X_4 > \nu_4) \]

\[ P(L \geq x) = \sum_{S \in \Omega_x} P(\text{assets in } S \text{ default}) \]
Normal Probabilities $\rightarrow$ Loss Probabilities $\rightarrow$ Prices

$$P(L \geq x) = \sum_{S \in \Omega_x} P(\text{assets in } S \text{ default})$$

So,

$$P_t(L \geq x) = a_0 + a_1 t + \frac{1}{2} a_2 t^2 + \cdots$$

Each $a_n$ a weighted sum of independent loss probs;

$$E_t(L - y)^+ = \int_{y}^{\infty} P_t(L > x) dx = \delta_0 + \delta_1 t + \frac{1}{2} \delta_2 t^2 + \cdots$$

Each $\delta_n$ a weighted sum of independent tranche prices

Tradeoff: Many 0-factor prices vs. one multifactor price
Example: 5-Factor Model

- Nonzero correlations are 0.2
- Tranche price with 16% attachment
- 1st, 2nd, and 3rd-order approximations
Extension: Expand From Single-Factor Model

• Expansion from independent model works well for weak correlations
• For strong correlations, write

\[ R_t = tR + (1-t)C \]

with
\[ R = \text{target correlation matrix} \]
\[ C = \text{single-factor correlation matrix} \]

Condition on single factor, expand, integrate over factor
Example

\[
\begin{pmatrix}
0.6 & 0.4 & 0.6 \\
0.6 & 0.4 & 0.6 \\
0.6 & 0.4 & 0.6 \\
0.6 & 0.4 & 0.6 \\
0.6 & 0.4 & 0.6 \\
0.6 & 0.4 & 0.6 \\
0.6 & 0.4 & 0.6 \\
0.6 & 0.4 & 0.6 \\
\end{pmatrix}
\]

- 200 names
- Blocks of 25
- Default probabilities 2%-4%
- 7 factors
- Expand around model with first factor only
Tranche Price, Linear and Quadratic Approx

\(E(L - y)^+ \) vs. \( y\)

**Linear**

**Quadratic**
Implied Correlation: 4-Factor Model

The graph illustrates the relationship between the fraction of total loss and the implied correlation using a 4-Factor Model. The curve shows how the implied correlation changes as the fraction of total loss increases. The graph includes a dotted line representing the model and a solid line representing the 1st-order approximation.
Quadratic Transform Approximation
Using the Characteristic Function

- $L$ = portfolio loss at a coupon date
- If we knew characteristic function

$$\phi(\omega) = E[\exp(i\omega L)], \quad i = \sqrt{-1}$$

we could invert numerically to find loss distribution

- Defaults are conditionally independent given factors, so we know the *conditional* char. fn.

$$\phi(\omega, Z) = E[\exp(i\omega L) | Z] = \prod_{k=1}^{N} \left(1 + p_k(Z)(1 - e^{i\omega c_k})\right)$$

$p_k(Z) = \text{conditional default probability, given } Z$
From Conditional to Unconditional Transform

• In a single-factor model, we can integrate out the factor numerically to get the \textit{unconditional} transform

\[
(2\pi)^{-1/2} \int E[\exp(i\omega L) | Z = z] e^{-z^2/2} dz
\]

• In a multifactor model, this again requires multidimensional integration

• Instead of doing
  – \textit{approximate integration of exact conditional transform}

we will do
  – \textit{exact integration of approximate conditional transform}
Quadratic Transform Approximation

Key property: An expectation of the form

\[ E[\exp(b'Z + Z'CZ)] = \frac{1}{\sqrt{\det(I - 2C)}} \exp\left(\frac{1}{2}b'(I - 2C)b\right) \]

can be evaluated in closed form, regardless of the dimension of \( Z \sim N(0, I) \)

So, we approximate

\[ \phi(\omega, Z) = E[\exp(i\omega L) \mid Z] = \prod_{k=1}^{N} \left(1 + p_k(Z)(1 - e^{i\omega_k})\right) \]

by an exponential quadratic function and take (exact) expectation over \( Z \) to get approximate characteristic function
Quadratic Transform Approximation

Key property:

\[ E[\exp(b'Z + Z' CZ)] = \frac{1}{\sqrt{\det(I - 2C)}} \exp\left(\frac{1}{2} b'(I - 2C)b\right) \]

regardless of the dimension of \( Z \sim N(0, I) \)

So, we approximate

\[ \phi(\omega, Z) = E[\exp(i\omega L) \mid Z] = \prod_{k=1}^{N} \left(1 + p_k(Z)(1 - e^{i\alpha_k})\right) \]

by an exponential quadratic function and take (exact) expectation over \( Z \) to get approximate characteristic function
Implied Correlation: 4-Factor Model

Approximation

Model

fraction of total loss

implied correlation
Summary

• Multifactor extensions of Gaussian copula useful in
  – Fitting implied correlation skew
  – Credible measurement of portfolio credit risk
• With more than 1 or 2 factors, numerical evaluation becomes a challenge: high-dimensional integration requires Monte Carlo
• We develop two fast and accurate approximations:
  – Correlation expansion around zero-factor or single-factor model
  – Quadratic transform approximation with an arbitrary number of factors