Pricing Credit Derivatives and Measuring Credit Risk in Multifactor Models

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Objective

- Develop fast, accurate approximations for
  - measuring credit risk
  - pricing credit derivatives

in multifactor Gaussian copula models

- Why multifactor Gaussian copula?
  - Gaussian copula is the industry standard
  - Skew in implied correlation contradicts single factor
  - For risk management, number of factors should be close to number of factors in an equity model

- Obstacles are calibration and computation

Risk Measurement and Pricing

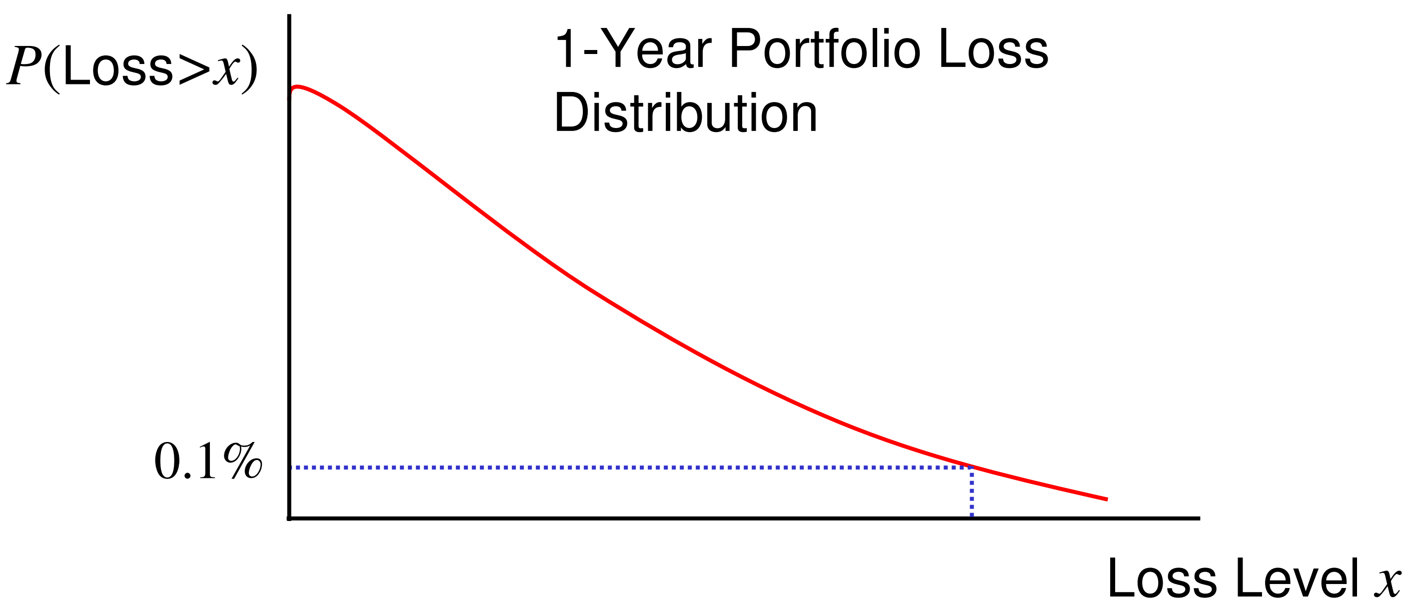
- Risk measurement problem:
  - Find the distribution of losses in a credit portfolio over a fixed horizon; calculate risk measure(s)

- Pricing problem
  - Price a CDO tranche
  - Price depends on the distribution of losses in the underlying collateral at each coupon date

In both cases, modeling the distribution of losses is key

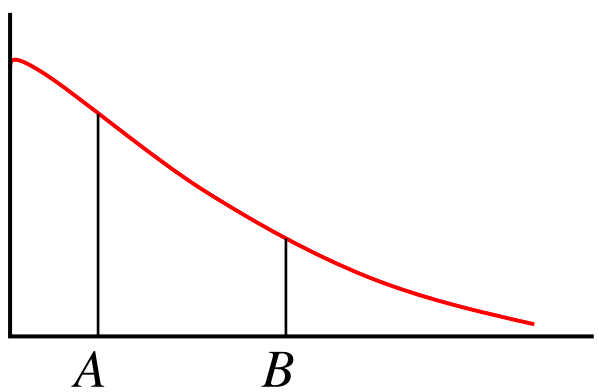
Risk Measurement and Pricing

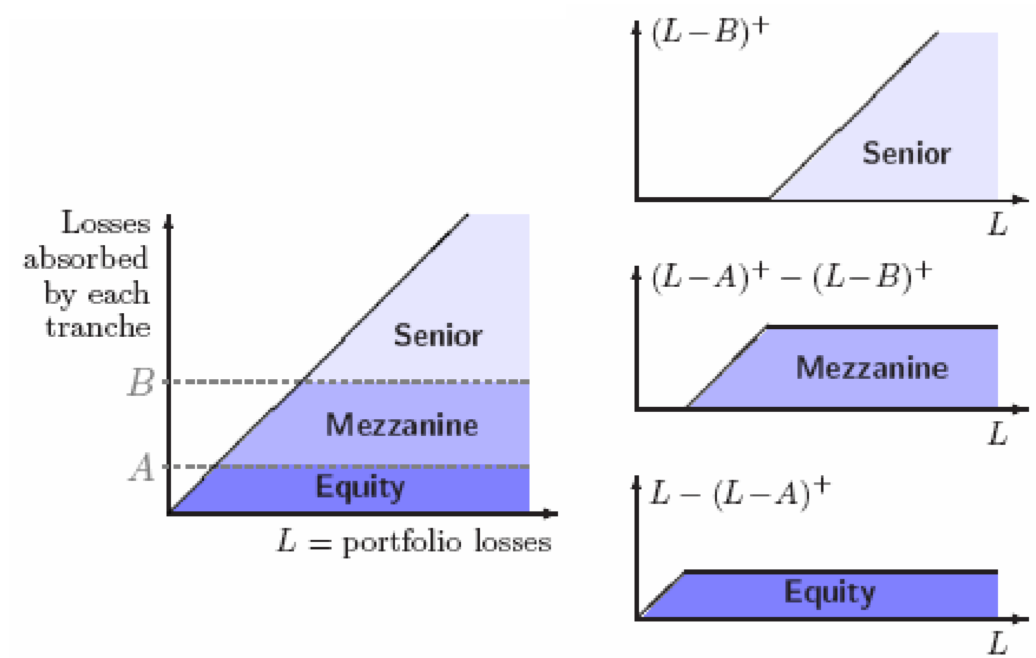
- Risk measurement problem:
  - Find the distribution of losses in a credit portfolio over a fixed horizon; calculate risk measure(s)



Risk Measurement and Pricing

- CDO Pricing problem:
  - Find the distribution of losses in a credit portfolio over a fixed horizon; calculate option-like payoff





For both problems, modeling the loss distribution is key

Outline

- Gaussian copula model of default times
- Multifactor models – advantages and difficulties
- Approximations
  - Correlation expansion: use a Taylor expansion around a zero-factor or single-factor model
  - Quadratic transform approximation: approximate the characteristic function of a multifactor model and invert the approximation

Modeling Default Times: Gaussian Copula

Modeling Default Times

In order to

- measure portfolio credit risk
- price portfolio credit derivatives

need to model the joint distribution of the default times $(\tau_1, \ldots, \tau_m)$ of the assets in the portfolio

Copula perspective:

i. specify the marginal distribution for each asset
   - use information from credit spreads, ratings

ii. specify the *dependence* ("correlation") between default times – this is the copula

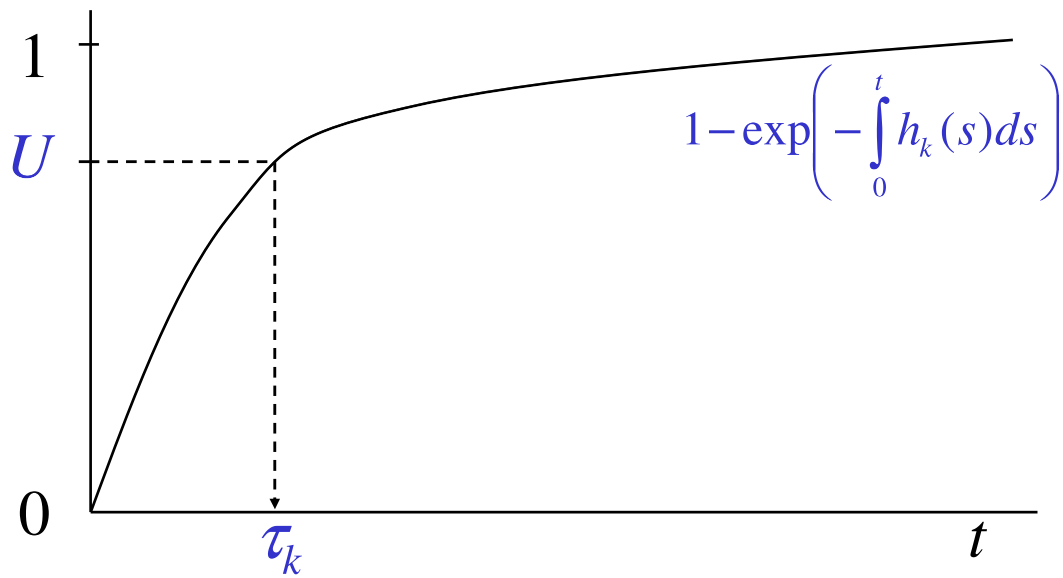
Default Times: Marginal Distributions

τ_k = time to default for name k in underlying portfolio

- Default time τ_k determined by *hazard rate function* for that name, $h_k(t)$
- This is, e.g., the CDS spread for that name
- Distribution of time to default for name k:

$$P(\tau_k \leq t) = 1 - \exp\left(-\int_0^t h_k(s)ds \right)$$

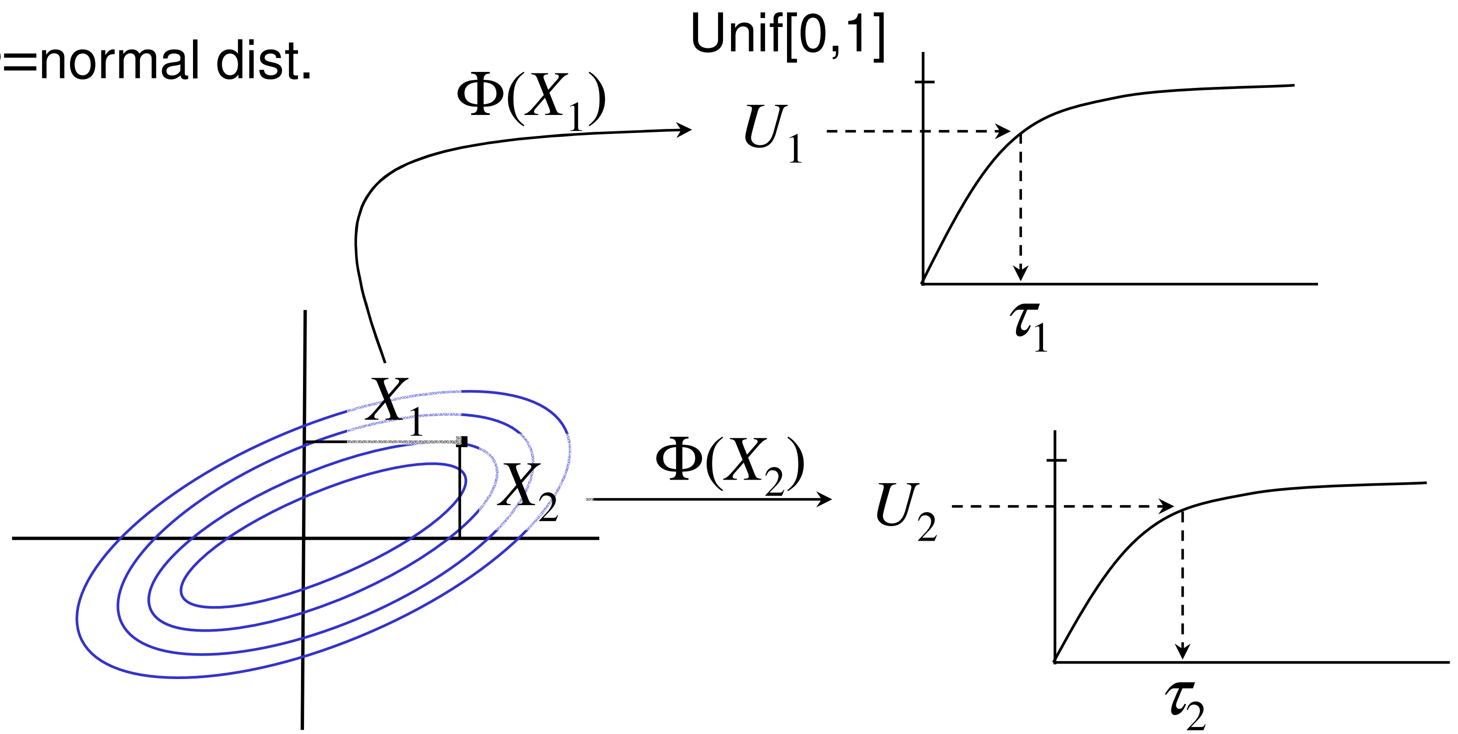
Simulating Default Time



- Generate Uniform[0,1] U
- Apply inverse distribution to produce default time τ_k

Gaussian Copula: Joint Distribution

Φ=normal dist.



Correlations in latent variables determine dependence between default times

Dependent Defaults: Gaussian Copula

$$\text{multivariate normal}\left\{\begin{array}{lll} X_1 \to & U_1 \to & \tau_1 \\ X_2 \to & U_2 \to & \tau_2 \\ \quad\vdots & \quad\vdots & \\ X_N \to & U_N \to & \tau_N \end{array}\right.$$

Cumulative normal Φ

Inverse distribution

$$(X_1, X_2, \ldots, X_N) \text{ is } N(0, \Sigma)$$

Dependence between default times $\tau_1,\ \tau_2,\ \ldots,\ \tau_N$
determined by correlation matrix Σ

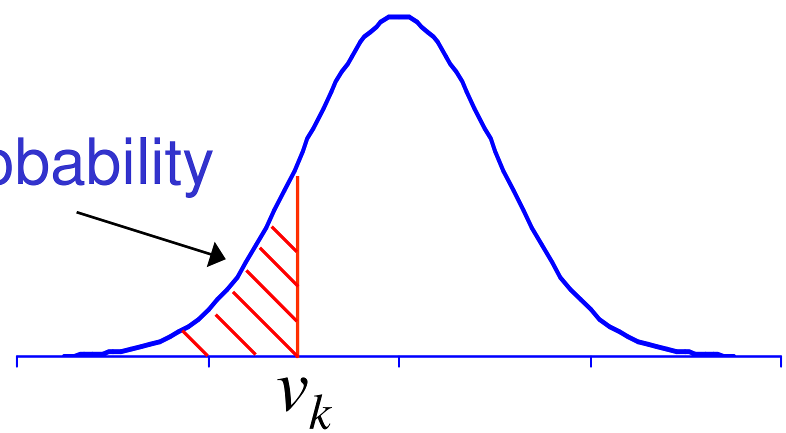
Losses at a Fixed Horizon

Consider losses at a fixed horizon T

kth asset defaults over this horizon if $\tau_k \le T$

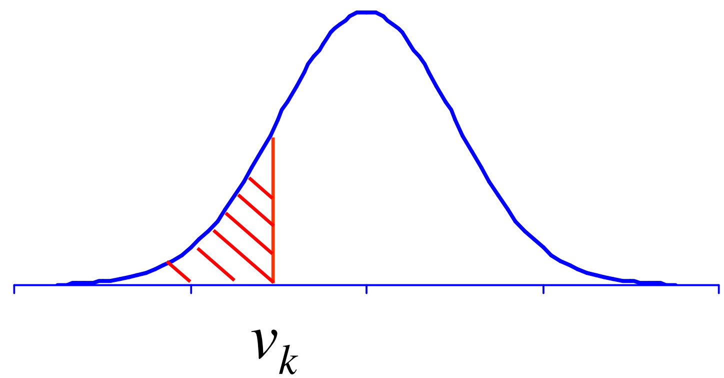
Equivalently, if its latent variable X_k falls below some threshold v_k

p_k=marginal default probability



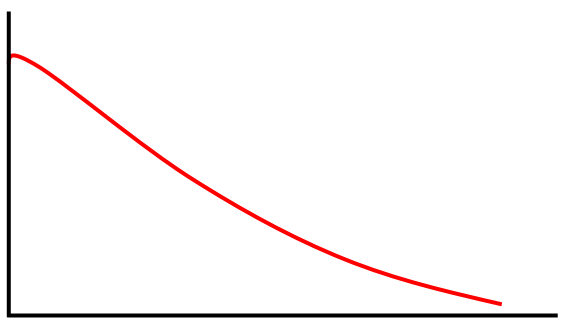
Loss Distribution at a Fixed Horizon

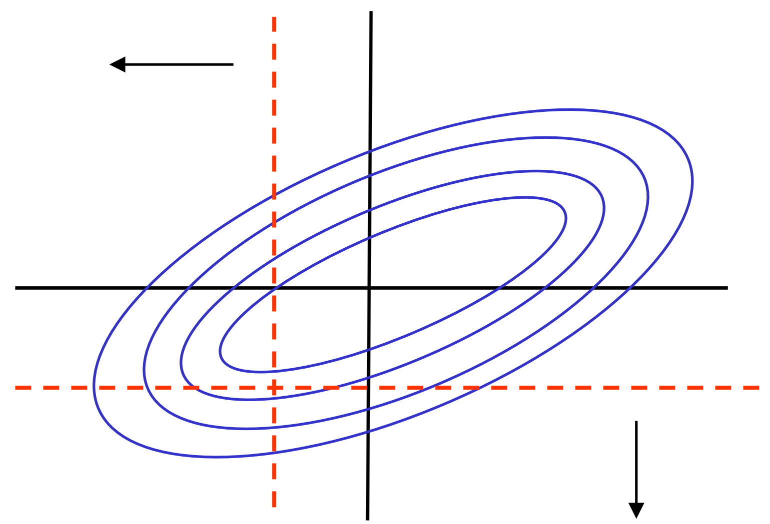
Loss given default c_k for kth asset



Portfolio loss at a fixed horizon is

$$L = \sum_{k=1}^{m} c_k \mathbb{1}\{X_k \le v_k\}$$





Once We Have the Portfolio Loss L…

…we can calculate VaR or some other portfolio risk measure

… we can price a CDO

Factor Models and Multifactor Models

Factor Models

- Recall the latent variables $(X_1,\ldots,X_N)$ have a multivariate normal distribution
- Factor representation

common factors

specific risk

$$X_k = a_{k1}Z_1 + \cdots + a_{kd}Z_d + b_k\varepsilon_k$$

factor loadings

- $Z_1,\ldots,Z_d,\ \varepsilon_k$ independent standard normals

Factor Models

- Latent variables loosely related to firm value through Merton model
- Factors may have economic interpretation

	Firm Value	Market	Energy	Auto	Telecomm	Europe	Japan	Specific Risk
Name 1	*X1*	*a11*	*a12*				*a1d*	*b1*
Name 2	*X2*	*a21*			*a24*	*a2,d-1*		*b2*
Name 3	*X3*	*a31*						*b3*
Name m	*Xm*	*am1*		*am3*			*amd*	*bm*

Factor Models

$$X_k = a_{k1}Z_1 + \cdots + a_{kd}Z_d + b_k\varepsilon_k$$

Loading matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{dd} \end{bmatrix}$$

Correlation matrix

$$AA' + \text{diag}(b_1^2, \ldots, b_m^2)$$

$$\rho_{ij} = \sum_{k=1}^{m} a_{ik}a_{kj}$$

Defaults become independent conditional on factors

Single-Factor Gaussian Copula

- Factor representation

$$X_k = \sqrt{\rho}Z + \sqrt{1-\rho}\varepsilon_k$$

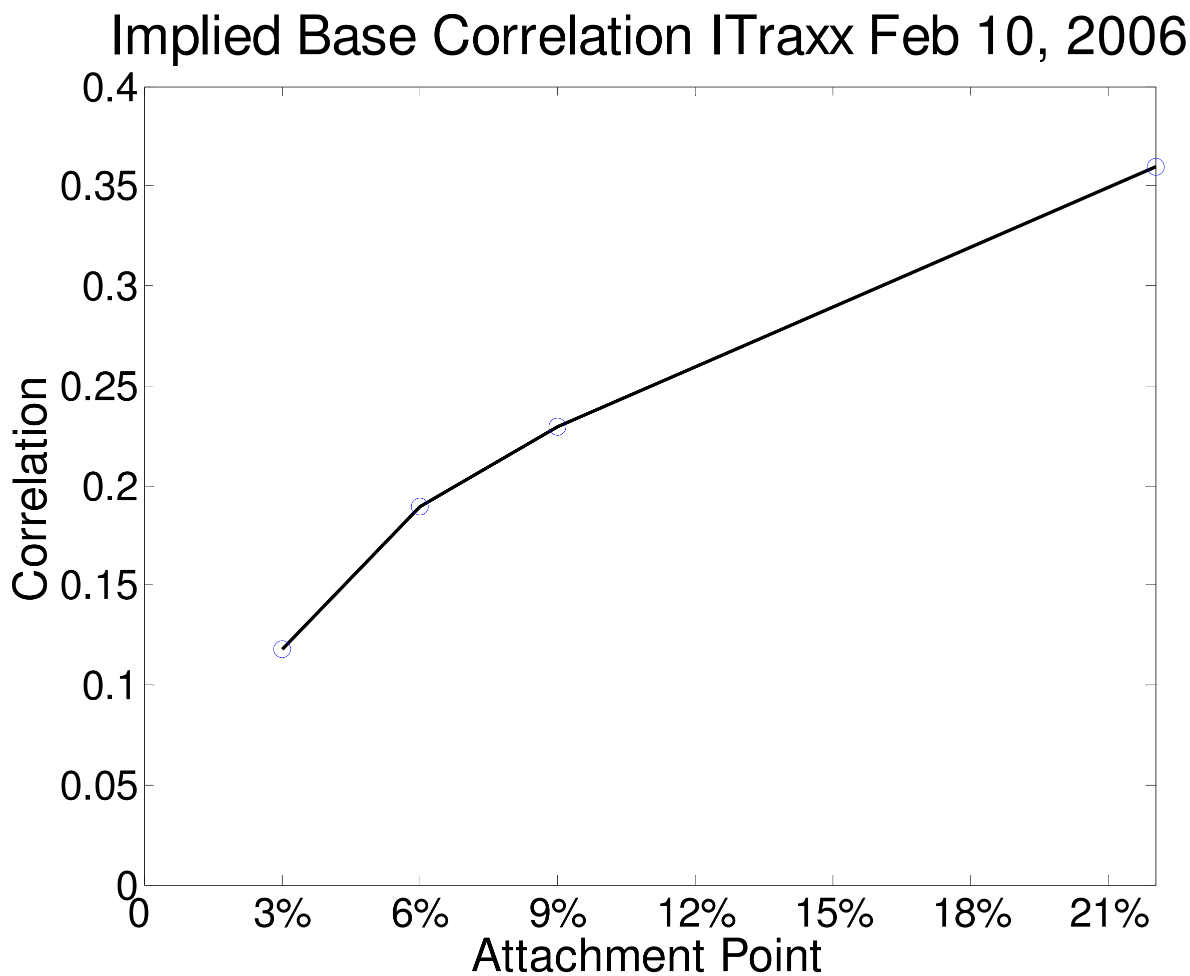
- Correlation matrix of $(X_1,\ldots,X_N)$

$$\begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & \ddots & & \vdots \\ \vdots & & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}$$

Implied Correlation Skew

Single-factor Gaussian copula can't match market data:

Different tranches priced at different correlations



Multifactor Models

- There are many ways of generating a skew
  - E.g., change the copula
- What about multiple factors?
- What kind of factor structure would generate the typical pattern of implied correlation?

- Need more correlation in rare defaults of highest quality credits

Example

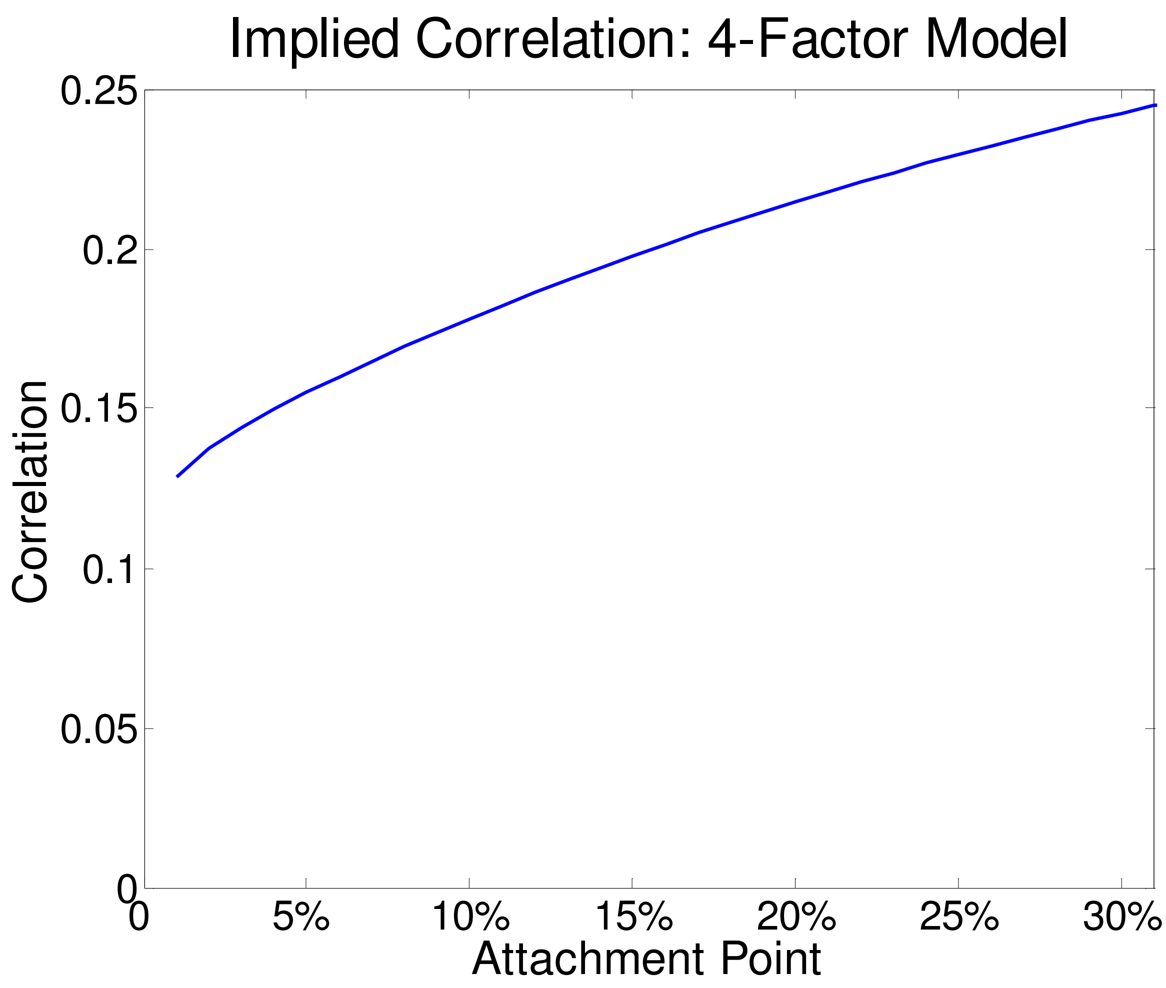
200 names in four blocks of 50 each

LGD = 1

$$\text{default prob.s} = \begin{bmatrix} 1\% \\ \vdots \\ 3\% \\ \vdots \\ 5\% \\ \vdots \\ 7\% \end{bmatrix}, \quad \text{factor loadings} = \begin{bmatrix} .3 & .4 & .5 & .6 \\ \vdots & \vdots & \vdots & \vdots \\ .3 & .4 & .5 & \\ \vdots & \vdots & & \\ .3 & .4 & & \\ \vdots & & & \\ .3 & & & \end{bmatrix}$$

50 highest-rated names have highest correlation

Implied Correlation: 4-Factor Model



Gaussian Copula Calculations

- "zero-factor" model (independent defaults) is tractable
- For a single-factor model:
  - Condition on the factor to make defaults independent
  - Integrate over the single factor
- Multifactor model?
  - Condition on factors to make defaults independent
  - Dimension of integration equals number of factors
  - When the number of factors is large, this requires Monte Carlo
- As an alternative, we develop approximations

Correlation Expansions

Correlation Expansions

- Given multifactor correlation matrix with entries ρ_{ij}
- Introduce family of correlation matrices with off-diagonal entries $t\rho_{ij}$, $0 \le t \le 1.$

$$
\underset{t=0}{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}}
\qquad
\begin{pmatrix} 1 & t\rho_{12} & \cdots & t\rho_{1d} \\ t\rho_{21} & 1 & & \vdots \\ \vdots & & \ddots & t\rho_{d-1,d} \\ t\rho_{d1} & \cdots & t\rho_{d,d-1} & 1 \end{pmatrix}
\qquad
\underset{t=1}{\begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1d} \\ \rho_{21} & 1 & & \vdots \\ \vdots & & \ddots & \rho_{d-1,d} \\ \rho_{d1} & \cdots & \rho_{d,d-1} & 1 \end{pmatrix}}
$$

independent model

target model

•Approximate target by expanding in t

Correlation Expansions

- Let L = portfolio loss at a coupon date
- E_t = expectation using correlations $t\rho_{ij}$
- Tranche price reduces to calculating

$$E_t(L-y)^+, \quad \text{at } t=1$$

- Expansion

$$E_t(L-y)^+ = \delta_0 + \delta_1 t + \frac{1}{2}\delta_2 t^2 + \cdots$$

Coefficients δ_i can be calculated in independent-obligor model

Correlation Expansions

Kibble (1945): For multivariate normal $X_1,\ldots,X_m$ with correlations $t\rho_{ij}$

$$P_t(X_1 > x_1,\ldots,X_m > x_m) = b_0 + b_1 t + \frac{1}{2} b_2 t^2 + \cdots$$

$$b_n = \sum_{\substack{1\le j_1<k_1\le m \\ \vdots \\ 1\le j_n<k_n\le m}} \rho_{j_1k_1} \cdots \rho_{j_nk_n} B_{\#1}(x_1)\cdots B_{\#m}(x_m)$$

$B_n(x) = \phi(x) H_{n-1}(x)$, H_k Hermite polynomials

We develop efficient method to calculate coefficients as weighted sum of independent probabilities

Approximating the Loss Distribution

Example: Suppose LGDs are 1, 2, 3, and 4

$$P(L \geq 10) = P(X_1 > v_1, X_2 > v_2, X_3 > v_3, X_4 > v_4)$$

$$P(L \geq 9) = P(X_2 > v_2, X_3 > v_3, X_4 > v_4)$$

$$P(L \geq 7) = P(X_1 > v_1, X_2 > v_2, X_3 \leq v_3, X_4 > v_4) + P(X_3 > v_3, X_4 > v_4)$$

$$P(L \geq x) = \sum_{S \in \Omega_x} P(\text{assets in } S \text{ default})$$

Normal Probabilities → Loss Probabilities → Prices

$$P(L \geq x) = \sum_{S \in \Omega_x} P(\text{assets in } S \text{ default})$$

So,

$$P_t(L \geq x) = a_0 + a_1 t + \frac{1}{2} a_2 t^2 + \cdots$$

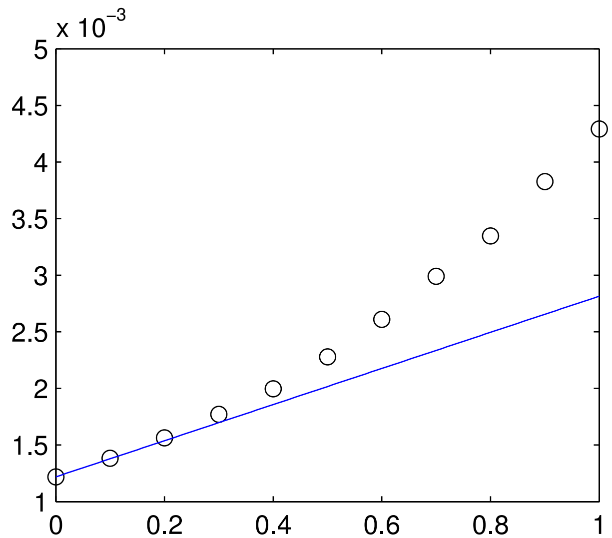
Each a_n a weighted sum of independent loss probs;

$$E_t(L - y)^+ = \int_y^\infty P_t(L > x)dx = \delta_0 + \delta_1 t + \frac{1}{2}\delta_2 t^2 + \cdots$$

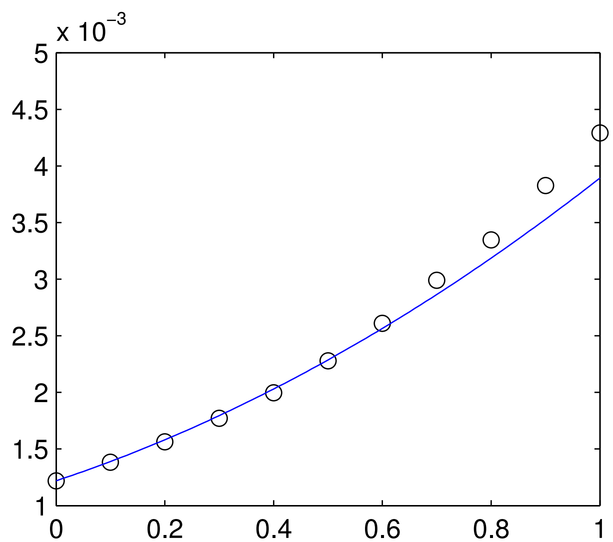
Each δ_n a weighted sum of independent tranche prices

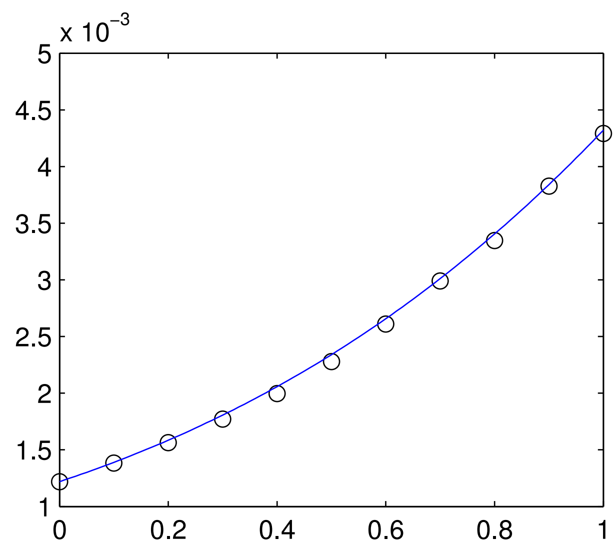
Tradeoff: Many 0-factor prices vs. one multifactor price

Example: 5-Factor Model



- Nonzero correlations are 0.2
- Tranche price with 16% attachment
- 1st, 2nd, and 3rd-order approximations





Extension: Expand From Single-Factor Model

- Expansion from independent model works well for weak correlations
- For strong correlations, write

$$R_t = tR + (1-t)C$$

with

R = target correlation matrix

C = single-factor correlation matrix

Condition on single factor, expand, integrate over factor

Example

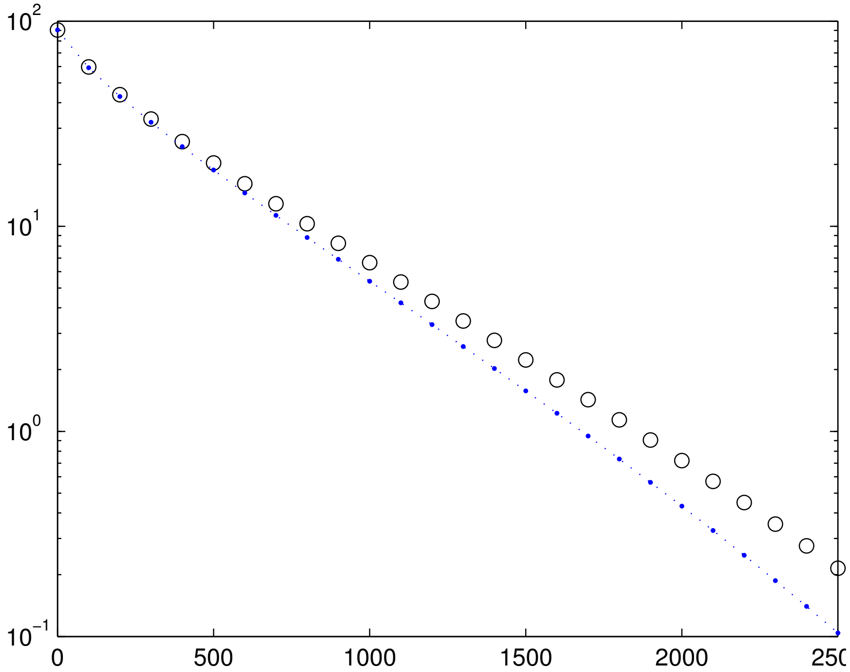
$$\begin{bmatrix} .6 & .4 & & & & .3 & \\ .6 & .4 & & & & & .3 \\ .6 & & .4 & & & .3 & \\ .6 & & .4 & & & & .3 \\ .6 & & & .4 & & .3 & \\ .6 & & & .4 & & & .3 \\ .6 & & & & .4 & .3 & \\ .6 & & & & .4 & & .3 \end{bmatrix}$$

- 200 names
- Blocks of 25
- Default probabilities 2%-4%
- 7 factors
- Expand around model with first factor only

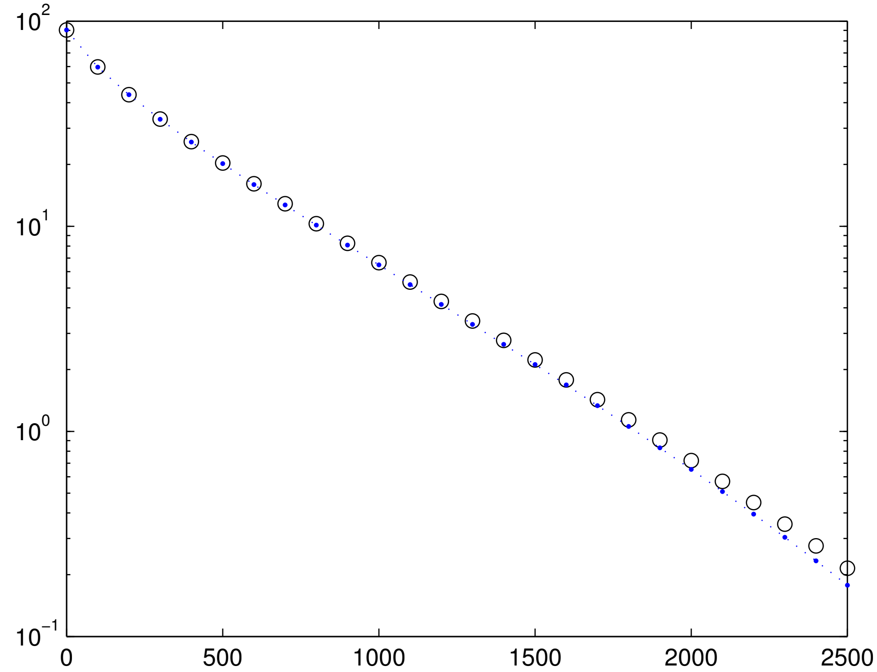
Tranche Price, Linear and Quadratic Approx

$$E(L-y)^{+} \text{ vs. } y$$

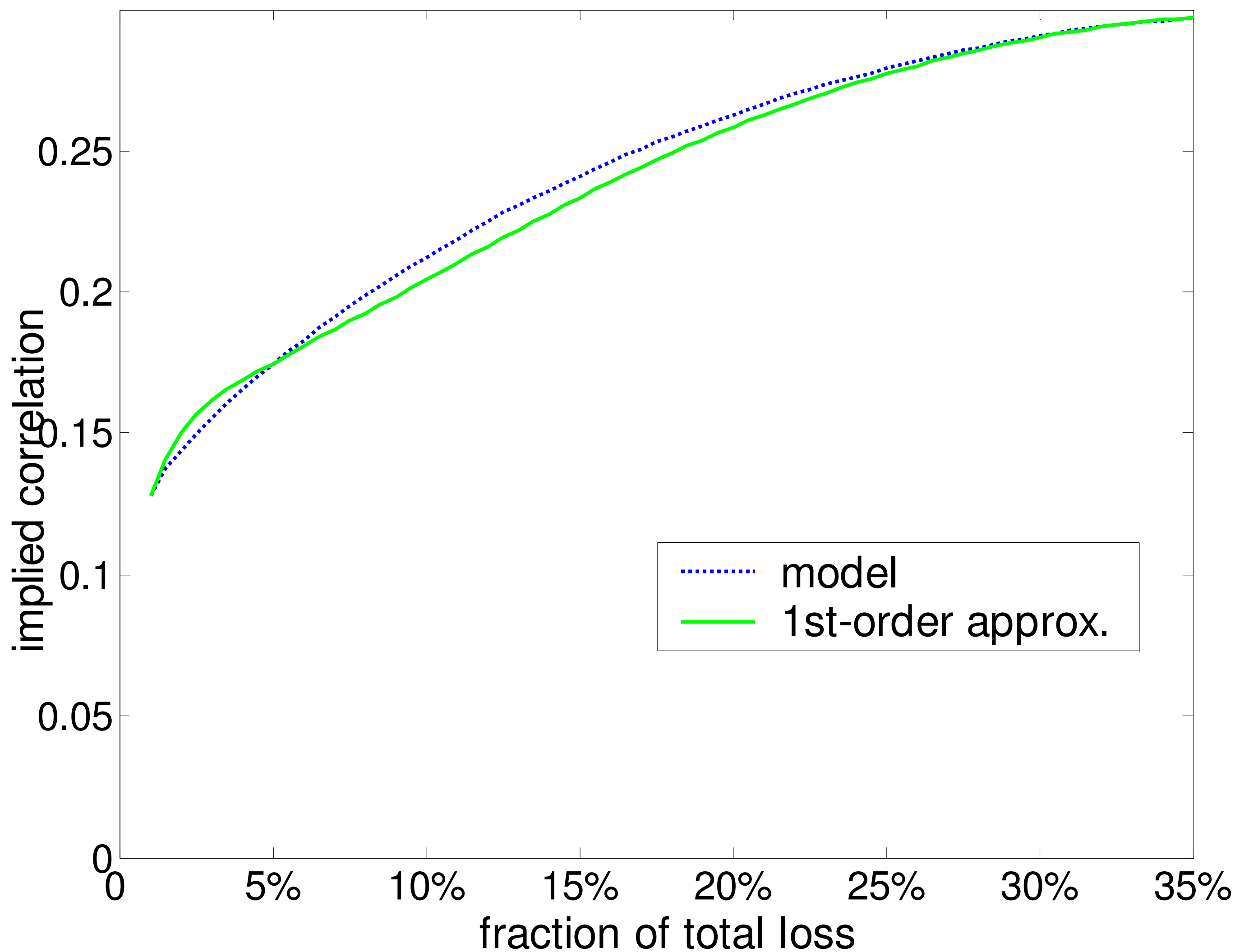
Linear



Quadratic



Implied Correlation: 4-Factor Model



Quadratic Transform Approximation

Using the Characteristic Function

- L = portfolio loss at a coupon date
- If we knew characteristic function

$$\phi(\omega) = E[\exp(i\omega L)], \qquad i = \sqrt{-1}$$

we could invert numerically to find loss distribution

- Defaults are conditionally independent given factors, so we know the <u>*conditional*</u> char. fn.

$$\phi(\omega, Z) = E[\exp(i\omega L) \mid Z] = \prod_{k=1}^{N} \left(1 + p_k(Z)(1 - e^{i\omega c_k}\right)$$

$p_k(Z)$ = conditional default probability, given Z

From Conditional to Unconditional Transform

- In a single-factor model, we can integrate out the factor numerically to get the *unconditional* transform

$$(2\pi)^{-1/2}\int E[\exp(i\omega L) \mid Z = z]e^{-z^2/2}dz$$

- In a multifactor model, this again requires multidimensional integration
- Instead of doing
  - *approximate integration of exact conditional transform*

we will do

  - *exact integration of approximate conditional transform*

Quadratic Transform Approximation

Key property: An expectation of the form

$$E\left[\exp\left(b'Z + Z'CZ\right)\right] = \frac{1}{\sqrt{\det(I-2C)}}\exp\left(\frac{1}{2}b'(I-2C)b\right)$$

can be evaluated in closed form, regardless of the dimension of $Z \sim N(0,I)$

So, we approximate

$$\phi(\omega, Z) = E[\exp(i\omega L) \mid Z] = \prod_{k=1}^{N}\left(1 + p_k(Z)(1-e^{i\omega c_k}\right)$$

by an exponential quadratic function and take (exact) expectation over Z to get approximate characteristic function

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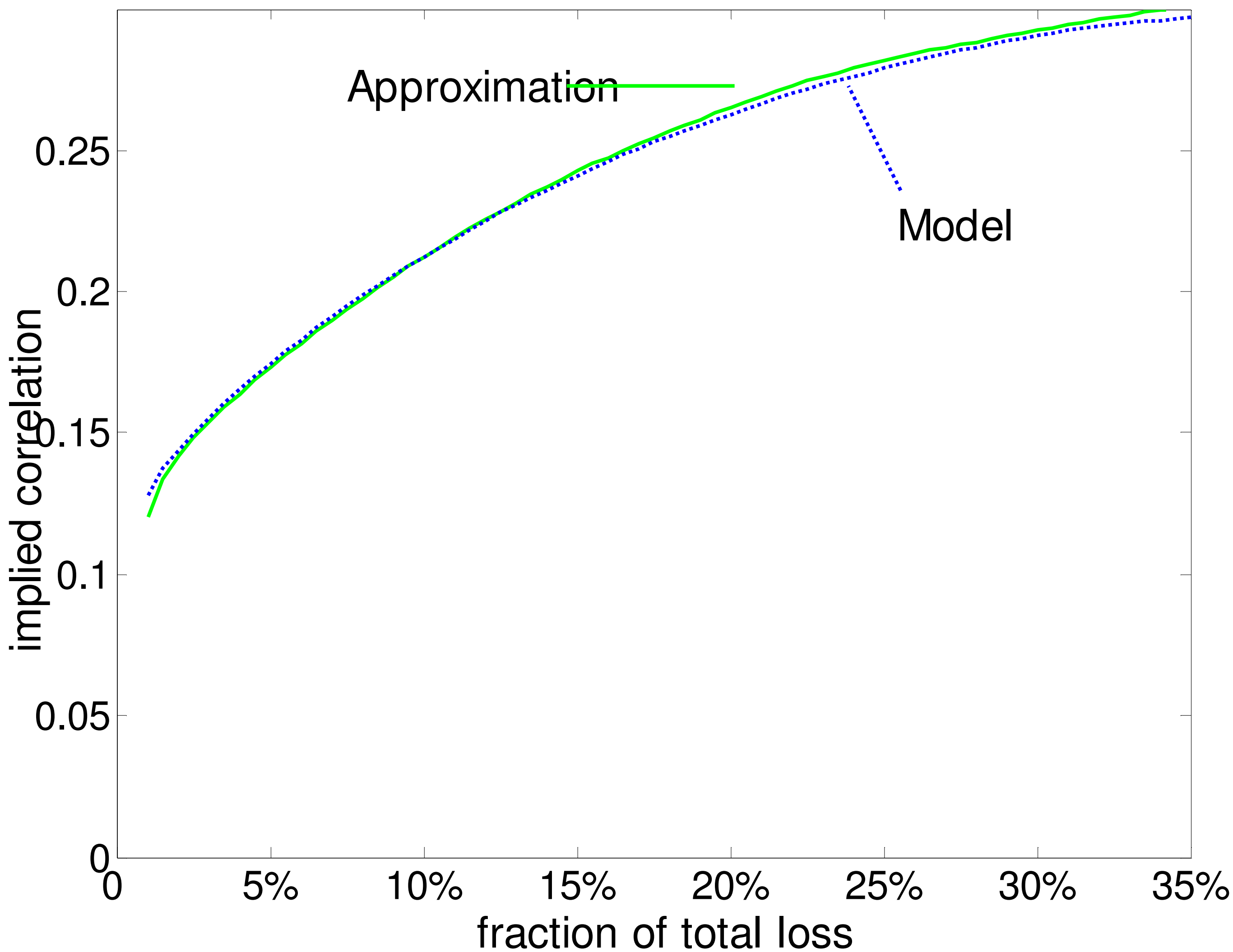
Implied Correlation: 4-Factor Model



Summary

- Multifactor extensions of Gaussian copula useful in
  - Fitting implied correlation skew
  - Credible measurement of portfolio credit risk
- With more than 1 or 2 factors, numerical evaluation becomes a challenge: high-dimensional integration requires Monte Carlo
- We develop two fast and accurate approximations:
  - Correlation expansion around zero-factor or single-factor model
  - Quadratic transform approximation with an arbitrary number of factors