Different Models of Default Risk Pricing and Hedging

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Begin at the beginning, and go on till you come to the end. ***Then, ...***

We consider, in a financial market

- A default event at some random time τ
- A **promised contingent claim** X, paid at time T if $\tau > T$
- A **recovery process** R: R_τ is the recovery payoff at time of default, if it occurs prior to or at the maturity date T.

We shall denote by

- $H_t = 1\!\!1_{\tau \leq t}$ the default process,
- $B(t,T)$ the price of a default-free zero-coupon bond with maturity T
- $B_t = \exp - \int_0^t r_s ds$ the discounted factor
- $B_T^t = \exp - \int_t^T r_s ds$.

A **Defaultable zero-coupon bond** (DZC) of maturity T pays

- 1 monetary unit at time T if the default has not occured
- 0 otherwise (i.e., if $\tau \leq T$).

Its price at time t is $D(t, T)$.

A **corporate bond with recovery R** pays

- 1 monetary unit at time T if $T < \tau$
- R_t at time t if the default occurs at time t, for $t < T$.

Reduced Form Approach

In the reduced form model, the knowledge if whether or not the default has occurred before time t is not included in the observation of the primary market.

The most popular model if when the default time is the first hitting time of a random barrier.

Other models are when the agent has only partial information on the prices.

We assume that there exists a primary financial market, with information $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$.
We denote by $\mathbf{G}$ the filtration which consists of

- the filtration of the default free market $\mathbf{F}$
- at time t the knowledge if whether or not, the default has occurred before time t.

Toy model

The default time is independent of the prices.

We denote by F^P the cumulative function of τ, and G^P the survival probability i.e.

$$F^P(t) \stackrel{def}{=} \mathbb{P}(\tau \leq t),\ G^P(t) \stackrel{def}{=} \mathbb{P}(\tau > t)$$

More interesting is the knowledge of the probability of the occurrence of the default, given that the default has not occurred before time t

$$\mathbb{P}(\tau > T | \tau > t) = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)} = \frac{G^P(T)}{G^P(t)}$$

We introduce $\Gamma^P(t) \stackrel{def}{=} -\ln(G^P(t))$ the **hazard function**, an increasing function assumed to be differentiable, i.e. of the form $\Gamma^P(t) = \int_0^t \gamma^P(s)ds$.

The process

$$\mathbb{1}_{\tau \leq t} - \Gamma^P(t \wedge \tau) = H_t - \int_0^{t \wedge \tau} \gamma^P(s)ds = H_t - \int_0^t (1 - H_s)\gamma^P(s)ds$$

where

$$\gamma^P(s) = \frac{f^P(s)}{1 - F^P(s)} = \mathbb{P}(\tau \in [s, s + ds] | \tau > s)$$

is a $\mathbb{P}$-martingale.

Of course, the hazard function depends on the choice of the probability, and the process

$$H_t - \int_0^{t \wedge \tau} \gamma^Q(s) ds$$

where

$$\gamma^Q(s) = \frac{f^Q(s)}{1 - F^Q(s)} = \frac{\mathbb{Q}(\tau \in ds)}{ds\, \mathbb{Q}(\tau > s)}$$

is a $\mathbb{Q}$- martingale (as soon as τ is still independent of $\mathcal{F}_\infty$).

In that setting, the market is incomplete.

If a DZC of maturity T is traded in the market at price $D(t,T)$, then the risk neutral probability can be deduced from these prices.

Indeed, from the fundamental theorem of asset pricing, the value of a DZC before the default time is

$$D(t,T) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{T<\tau} B_T^t | \mathcal{F}_t \vee \tau > t) = \frac{G(T)}{G(t)} B(t,T)$$

and obviously 0 after the default. Here $\mathbb{Q}$ is the pricing probability, chosen by the market and $G(t) = \mathbb{Q}(t < \tau)$. The price of a DZC is

$$D(t,T) = \mathbb{1}_{t<\tau} \frac{G(T)}{G(t)} B(t,T)$$

The price of a corporate bond with deterministic recovery R is

$$D^{(R)}(t,T) = 1\!\!1_{t<\tau}\left(\frac{\mathbb{Q}(T<\tau)}{\mathbb{Q}(t<\tau)}B(t,T) + \frac{1}{\mathbb{Q}(t<\tau)}\int_t^T B(t,s)R(s)\mathbb{P}(\tau\in ds)\right)$$

The risk-neutral dynamics of a corporate bond is

$$dD^{(R)}(t,T) = \Big(r(t)D^{(R)}(t,T) - R(t)\gamma(t)(1-H_t)\Big)\,dt - \widetilde{D}^{(R)}(t,T)dM_t$$

where M is the risk-neutral martingale $M_t = H_t - \int_0^t (1-H_s)\gamma(s)ds$.

In the case of null interest rate the value at time t of a defaultable payoff $h(S_T)$ is, if the default has not occurred

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\tau>T}h(S_T)|\mathcal{G}_t) = \frac{\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\tau>T}h(S_T)|\mathcal{F}_t)}{\mathbb{Q}(\tau > t)}$$

In our model, the independence assumption implies that

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\tau>T}h(S_T)|\mathcal{F}_t) \quad = \quad \mathbb{Q}(\tau > T)\mathbb{E}_{\mathbb{Q}}(h(S_T)|\mathcal{F}_t) = G_T\mathbb{E}_{\mathbb{Q}}(h(S_T)|\mathcal{F}_t)$$

In the general case

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\tau>T}h(S_T)B_T|\mathcal{G}_t) = \mathbb{1}_{t<\tau}\mathbb{E}_{\mathbb{Q}}\left(h(S_T)\exp(-\int_t^T(r(s)+\gamma(s))ds)|\mathcal{F}_t\right)$$

The (discounted) price of a defaultable claim is

$$\widehat{h}_t = \mathbb{E}_{\mathbb{Q}}(1\!\!1_{\tau>T} h(S_T) B_T^t | \mathcal{G}_t) = D(t,T)\mathbb{E}_{\mathbb{Q}}(h(S_T) B_T^t | \mathcal{F}_t)$$

Assuming that a DZC is traded in the market, as well as the underlying asset S and that the underlying default free market is complete, an hedging strategy for the defaultable claim is **to invest an amount $\widehat{h}_t$ in the DZC**.

Proof in the case $r = 0$:

Indeed, the price of the defaultable claim is

$$\widehat{h}_t = D(t,T)V_t = \mathbb{1}_{t<\tau} G(t)^{-1} G(T) \mathbb{E}_{\mathbb{Q}}(h(S_T)|\mathcal{F}_t)$$

hence, denoting by Δ the hedging strategy for the default free contingent claim $h(S_T)$, i.e. the process such that

$$h(S_T) = h + \int_0^T \Delta_s dS_s$$

$$d\widehat{h}_t = V_t dD(t,T) + D(t,T)dV_t = V_t dD(t,T) + D(t,T)\Delta_t dS_t$$

Credit default swap

The price at time $t \in [s, T]$ of a credit default swap started at s, with rate κ and protection payment $\delta(\tau)$ at default, equals

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \left(- \int_t^T \delta(u)\, dG(u) - \kappa \int_t^T G(u)\, du \right).$$

The dynamics of the price $S_t(\kappa)$, $t \in [s, T]$, are

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt.$$

Correlations

Let us consider the case of two defaults and $H^i_t = \mathbb{1}_{\tau_i \le t}$. Let

$$\mathbb{Q}(\tau_1 > t_1, \tau_2 > t_2) = G(t_1, t_2)$$

Let S^1 be the price of a CDS written on τ_1. On the set $\{\tau_1 > t, \tau_2 > t\}$

$$\begin{aligned} S^1_t &= \frac{1}{G(t,t)} \left(- \int_t^T \delta_1(u) G(du, t) - \kappa_1 \int_t^T du\, G(u,t) \right) \\ &= V^1(t) \end{aligned}$$

and, on the set $\{\tau_1 > t > \tau_2\}$

$$\begin{aligned} S^1_t &= \frac{1}{\partial_2 G(t, \tau_2)} \left(- \int_t^T du\, \delta_1(u) f(u, \tau_2) - \kappa_1 \int_t^T du\, \partial_2 G(u, \tau_2) \right) \\ &= V^2(t, \tau_2) \end{aligned}$$

It can be proved that

$$H_t^1 - \int_0^{t\wedge\tau_1\wedge\tau_2} \gamma_1(s)ds - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \gamma^{1|2}(s,\tau_2)ds$$

is a martingale with

$$\gamma_1(s) = -\frac{\partial_1 G(s,s)}{G(s,s)}, \qquad \gamma^{1|2}(t,s) = -\frac{f(t,s)}{\partial_2 G(t,s)}$$

Hence

$$S_t^1 = (1-H_t^1)(1-H_t^2)\, V^1(t) + (1-H_t^1)\, H_t^2\, V^2(t,\tau_2)$$

and

$$
\begin{aligned}
dS^1_t &= (1 - H^1_t)(1 - H^2_t)\, dV^1(t) + (1 - H^1_t)\, H^2_t\, dV^2(t, \tau_2) \\
&\quad - S^1_{t-}\, dH^1_t - (1 - H^1_t)\{V^1(t) - V^2(t, \tau_2)\}\, dH^2_t
\end{aligned}
$$

where

$$
\begin{aligned}
dV^1(t) &= \big((\gamma_1(t) + \gamma_2(t))\, V^1(t) + \kappa_1 - \delta_1(t)\gamma_1(t) - V^2(t,t)\gamma_2(t)\big)\; dt \\
dV^2(t, \tau_2) &= \Big(\gamma^{1|2}(t, \tau_2) V^2(t, \tau_2) + \kappa_1 - \gamma^{1|2}(t, \tau_2)\delta_1(t)\Big)\; dt
\end{aligned}
$$

Let us denote by $\widetilde{S}$ he predefault price, that is $S_t^1 = \mathbb{1}_{t<\tau} \widetilde{S}_t^1$

The pre-default price of a FtD claim $(X, 0, Z, \tau_{(1)})$, where $Z = (Z_1, Z_2)$ and $X = c(T)$, equals

$$\begin{aligned}
\widetilde{\pi}(t) &= \frac{\int_t^T du Z_1(u) \int_t^\infty dv\, f(u,v) + \int_t^\infty du \int_t^T dv Z_2(v)\, f(u,v)}{G(t,t)} \\
&\quad + c(T)\frac{G(T,T)}{G(t,t)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
d\widetilde{\pi}(t) &= \gamma(t)\widetilde{\pi}(t)\,dt - \sum_{i=1}^{n} Z_i(t)\gamma_i(t)\,dt, \\
&= \sum_{i=1}^{n} \gamma_i(t)\big(\widetilde{\pi}(t) - Z_i(t)\big)\,dt.
\end{aligned}$$

Assume that the linear system

$$\begin{aligned}
\phi^1_t\big(\widetilde{S}^1_t(\kappa_1)-\delta_1(t)\big)+\phi^2_t\big(\widetilde{S}^2_t(\kappa_2)-S^2_{t|1}(\kappa_2)\big) &= Z_1(t)-\widetilde{\pi}(t),\\
\phi^2_t\big(\widetilde{S}^2_t(\kappa_2)-\delta_2(t)\big)+\phi^1_t\big(\widetilde{S}^1_t(\kappa_1)-S^1_{t|2}(\kappa_1)\big) &= Z_2(t)-\widetilde{\pi}(t),
\end{aligned}$$

admits a unique solution $\phi_t=(\phi^1_t,\phi^2_t)$ and let

$$\phi^0_t=V_t(\phi)-\phi^1_tS^1_t(\kappa_1)-\phi^2_tS^2_t(\kappa_2)$$

where

$$dV_t(\phi)=\sum_{i=1}^{2}\phi^i_t\big(dS^i_t(\kappa_i)-\kappa_i dt\big)\,,\ V_0(\phi)=E_Q(Y)\,.$$

Then the self-financing strategy ϕ replicates the first-to-default claim $(X,0,Z,\tau_{(1)})$.

Hazard process models

- A reference **filtration** $\mathbf{F}$
- A **random time** τ

are given

- $\mathbf{H}$ is the filtration generated by the process $H_t = \mathbb{1}_{\tau \leq t}$
- $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$

Seting $F_t = \mathbb{Q}(\tau \le t | \mathcal{F}_t) = 1 - G_t$ and $\Gamma_t = -\ln(1 - F_t)$, we obtain the key formulae:

for $X \in \mathcal{F}_T$,

$$\mathbb{E}_{\mathbb{Q}}(X 1\!\!1_{T<\tau} | \mathcal{G}_t) = 1\!\!1_{\tau > t} e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}}(e^{-\Gamma_T} X | \mathcal{F}_t) \, .$$

Let R be an **F**-predictable process. Then,

$$\mathbb{E}_{\mathbb{Q}}(R_\tau \mathbb{1}_{\tau<T}|\mathcal{G}_t) = R_\tau \mathbb{1}_{\{\tau<t\}} + \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{Q}}(\int_t^T R_u dF_u|\mathcal{F}_t)$$

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If R is deterministic

$$\mathbb{E}_{\mathbb{Q}}(R_\tau \mathbb{1}_{\tau<T}|\mathcal{G}_t) = R_\tau \mathbb{1}_{\{\tau<t\}} + \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t} \int_t^T R_u \mathbb{Q}(\tau \in du|\mathcal{F}_t)$$

Let $F_t = \mathbb{Q}(\tau \leq t|\mathcal{F}_t)$ assumed to be **continuous** and satisfying $F_t < 1$.

This submartingale admits a Doob-Meyer decomposition as

$$F_t = Z_t + A_t$$

where Z is an **$\mathbb{F}$-martingale** and A a **predictable increasing** process.

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Assuming that $A_t = \int_0^t a_s ds$ the process

$$M_t = H_t - \Lambda_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} \frac{a_s}{1-F_s} ds = H_t - \int_0^{t\wedge\tau} \lambda_s ds$$

is a **G**-martingale.

$$\mathbb{E}_{\mathbb{Q}}(X \mathbb{1}_{T<\tau} | \mathcal{G}_t) = \mathbb{1}_{\tau>t} \left(e^{\Lambda_t} \mathbb{E}_{\mathbb{Q}}(e^{-\Lambda_T} X | \mathcal{G}_t) - \mathbb{E}_{\mathbb{Q}}(e^{\Lambda_\tau} \Delta Y_\tau \mathbb{1}_{\tau<T} | \mathcal{G}_t) \right)$$

where $Y_t = \mathbb{E}_{\mathbb{Q}} \left(X \exp\left(-\Lambda_T\right) | \mathcal{G}_t \right)$.

The intensity rate λ can be obtained as

$$\lambda_t = \lim_{h \to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t + h | \mathcal{F}_t)}{\mathbb{P}(t < \tau | \mathcal{F}_t)} .$$

The supermartingale $G = 1 - Z - A$ admits a **multiplicative decomposition** $G_t = C_t N_t$ where N is a martingale and C a decreasing process satisfying

$$dN_t = -\frac{1}{C_t} dZ_t, \; dC_t = -C_t \frac{1}{G_t} dA_t \,.$$

Hence

$$C_t = \exp - \int_0^t \frac{1}{G_s} dA_s = \exp - \Lambda_t$$

and

$$e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}}(e^{-\Gamma_T} X | \mathcal{F}_t) = \widehat{\mathbb{E}_{\mathbb{Q}}}(X \frac{C_T}{C_t} | \mathcal{F}_t) = \widehat{\mathbb{E}_{\mathbb{Q}}}(X \exp(- \int_t^T \lambda_s ds) | \mathcal{F}_t)$$

where

$$d\widehat{Q} = L_t d\mathbb{Q}, \quad dL_t = - \exp(\Lambda_t) \, L_t \, dZ_t \,.$$

In the particular case where F is increasing and continuous, and if $\mathbb{Q}(\tau \leq t|\mathcal{F}_t) = \mathbb{Q}(\tau \leq t|\mathcal{F}_\infty)$, then the intensity is $\Lambda = \Gamma$ and there exists a r.v. Θ, independent of $\mathcal{F}_\infty$ such that

$$\tau = \inf\{t \,:\, \Lambda_t \geq \Theta\}$$

Information at discrete times

Assume that

$$dV_t = V_t(\mu dt + \sigma dW_t), \; V_0 = v$$

i.e., $V_t = ve^{\sigma(W_t+\nu t)} = ve^{\sigma X_t}$. The default time is assumed to be the first hitting time of α with $\alpha < v$, i.e.,

$$\tau = \inf\{t \,:\, V_t \le \alpha\} = \inf\{t \,:\, X_t \le a\}$$

where $a = \sigma^{-1}\ln(\alpha/v)$. Here, $\mathbf{F}$ is the filtration of the observations of V at discrete times $t_1, \cdots t_n$ where $t_n \le t < t_{n+1}$, i.e.,

$$\mathcal{F}_t = \sigma(V_{t_1}, \cdots, V_{t_n}, \, t_i \le t)$$

The process $F_t = P(\tau \leq t | \mathcal{F}_t)$ is **continuous and increasing in** $[t_i, t_{i+1}[$ but is **not neither continuous nor increasing** on $\mathbb{R}^+$. The price of a DZC is, for $t \in [t_i, t_{i+1}[$

$$B_d^1(t,T) = e^{-r(T-t)} \mathbb{1}_{t<\tau} \frac{\Phi(T-t_i, a-X_{t_i})}{\Phi(t-t_i, a-X_{t_i})}$$

where

$$\begin{aligned} \Phi(t,z) &= \mathcal{N}\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z}\mathcal{N}\left(\frac{z+\nu t}{\sqrt{t}}\right), \qquad \text{for } z<0,\ t>0, \\ &= 0, \qquad \text{for } z \geq 0,\ t \geq 0, \\ \Phi(0,z) &= 1, \qquad \text{for } z<0 \end{aligned}$$

The Doob-Meyer decomposition of F is

$$F_t = \zeta_t + (F_t - \zeta_t),$$

where

$$\zeta_t = \sum_{i, t_i \leq t} \Delta F_{t_i}.$$

is an **F**-**martingale** and $F_t - \zeta_t$ is a **predictable increasing process**.

From

$$\mathbb{P}(\inf_{s \le t} X_s > z) = \Phi(t, z)\,,$$

We obtain for $t_1 < t < t_2$ and $X_{t_1} > a$

$$F_t = 1 - \Phi(t - t_1, a - X_{t_1}) \left[1 - \exp\left(-\frac{2\,a}{t_1}\,(a - X_{t_1}) \right) \right].$$

The case $X_{t_1} \le a$ corresponds to default: for $X_{t_1} \le a$, $F_t = 1$.

Hedging in a general model

Two default free assets, one total default asset

We assume that 3 (non paying dividends) assets are traded, with price's dynamics

$$\begin{aligned} dY_t &= Y_t r dt \\ dY_t^2 &= Y_t^2(\mu_{2,t}dt + \sigma_{2,t}dW_t) \\ dY_t^3 &= Y_{t-}^3(\mu_{3,t}dt + \sigma_{3,t}dW_t - dM_t)\,. \end{aligned}$$

Note that $Y_t^3 = 0$ on the set $t \geq \tau$ and that the predefault price follows

$$d\widetilde{Y}_t^3 = \widetilde{Y}_t^3((\mu_{3,t} + \gamma_t)dt + \sigma_{3,t}dW_t)$$

There exists a unique e.m.m. as soon as $\sigma_1 \neq \sigma_2$ and the risk-neutral intensity γ^* satisfies

$$\gamma^* = \gamma + \mu_3 - r + (r - \mu_2)\frac{\sigma_2}{\sigma_3} > 0$$

The hedging strategy of the defaultable claim $(X, 0, \tau)$ consists of a triple ϕ^1, ϕ^1, ϕ^3 such that

$$\begin{aligned} \phi_t^3 Y_t^3 &= C_t^{X,0} = 1\!\!1_{t<\tau} e^{-r(T-t)} \, \mathbb{E}_{\mathbb{Q}}(X 1\!\!1_{T<\tau} \,|\, \mathcal{G}_t), \\ \phi_t^1 e^{rt} + \phi_t^2 Y_t^2 &= 0 \end{aligned}$$

and which satisfies the self financing condition

$$dC_t = \phi_t^1 e^{rt} r dt + \phi_t^2 dY_t^2 + \phi_t^3 dY_t^3$$

If **the recovery payement is done at default time**, the hedging strategy the defaultable claim $(0, R, \tau)$ consists of a triple ϕ^1, ϕ^1, ϕ^3 such that

$$\phi_t^3 Y_t^3 = C_t^{0,R} - R_t, \quad \phi_t^1 e^{rt} + \phi_t^2 Y_t^2 = R_t, \quad t \leq \tau$$

$$\phi_t^i = 0, \quad t > \tau$$

and which satisfies the self financing condition.

If the **payement is done at maturity**, the hedging strategy consists of a triple ϕ^1, ϕ^1, ϕ^3 such that

$$\begin{aligned} \phi_t^3 Y_t^3 &= C_t^{0,Z} - R_t e^{-r(T-t)} \\ \phi_t^1 e^{rt} + \phi_t^2 Y_t^2 &= R_t e^{-r(T-t)}, \quad t \leq \tau \end{aligned}$$

$$\phi_t^3 = \phi_t^2 = 0, \quad \phi_t^1 = R_\tau e^{-r(T-\tau)}, \quad t > \tau$$

and which satisfies the self financing condition.

Sketch of the proof: Let us denote by $\widetilde{Y}^3$ the **F**-adapted version of the predefault price of Y^3:

$$Y_t^3 = 1\!\!1_{t<\tau} \widetilde{Y}_t^3$$

Let $\phi = (\phi^1, \phi^2, \phi^3)$ be a self-financing strategy satisfying **the balance condition** $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0$, and let $V_t(\phi) = \sum_{i=1}^{3} \phi_t^i Y_t^i = \phi_t^3 Y_t^3$ be the value of this strategy. The predefault prices in the numeraire $\widetilde{Y}^3$ are

$$\widetilde{V}_t^3(\phi) \quad = \quad \frac{\widetilde{V}_t(\phi)}{\widetilde{Y}_t^3}, \quad \widetilde{Y}_t^{i,3} = \frac{Y_t^i}{\widetilde{Y}_t^3}, \quad i = 1, 2,$$

Introduce the synthetic asset

$$d\widetilde{Y}_t^* \quad = \quad d\widetilde{Y}_t^{2,3} - \frac{\widetilde{Y}_t^{2,3}}{\widetilde{Y}_t^{1,3}}\, d\widetilde{Y}_t^{1,3} \,.$$

Then the pre-default wealth process $\widetilde{V}(\phi)$ satisfies

$$\widetilde{V}_t(\phi) = \widetilde{Y}_t^3 \left(\widetilde{V}_0^3(\phi) + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^* \right) .$$

The converse holds true.

PDE Approach

We are working in a model with constant (or Markovian) coefficients

$$
\begin{aligned}
dY_t &= Y_t r dt \\
dY_t^2 &= Y_t^2(\mu_2 dt + \sigma_2 dW_t) \\
dY_t^3 &= Y_{t-}^3(\mu_3 dt + \sigma_3 dW_t - dM_t)\,.
\end{aligned}
$$

Let $C(t, Y_t^2, Y_t^3, H_t)$ be the price of the contingent claim $G(Y_T^2, Y_T^3, H_T)$ and γ^* be the risk-neutral intensity of default.

The process $e^{-rt}C(t, Y_t^2, Y_t^3, H_t)$ is a martingale under the e.m.m.

Then,

$$\partial_t C(t, y_2, y_3; 0) + r y_2 \partial_2 C(t, y_2, y_3; 0) + r^* y_3 \partial_3 C(t, y_2, y_3; 0) - r^* C(t, y_2, y_3; 0)$$
$$+ \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 0) + \gamma^* C(t, y_2, 0; 1) = 0$$

where $r^* = r + \gamma^*$ and

$$\partial_t C(t, y_2; 1) + r y_2 \partial_2 C(t, y_2; 1) + \frac{1}{2} \sigma_2^2 y_2^2 \partial_{22} C(t, y_2; 1) - r C(t, y_2; 1) = 0$$

with the terminal conditions

$$C(T, y_2, y_3; 0) = G(y_2, y_3; 0), \quad C(T, y_2; 1) = G(y_2, 0; 1).$$

The ***replicating strategy*** ϕ *for* Y *is given by formulae*

$$
\begin{aligned}
\phi_t^3 Y_{t-}^3 &= -C(t, Y_t^2, 0; 1) + C(t, Y_t^2, Y_{t-}^3; 0) = -\Delta C(t) \\
\sigma_2 \phi_t^2 Y_t^2 &= -\Delta C(t) + \sum_{i=2}^{3} Y_{t-}^i \sigma_i \partial_i C(t) \\
\phi_t^1 Y_t^1 &= C(t) - \phi_t^2 Y_t^2 - \phi_t^3 Y_t^3 .
\end{aligned}
$$

Note that, in the case of survival claim, $C(t, Y_t^2, 0; 1) = 0$ and $\phi_t^3 Y_{t-}^3 = C(t, Y_{t-}^2, Y_{t-}^3; 0)$ for every $t \in [0, T]$. Hence, the following relationships holds, for every $t < \tau$,

$$\phi_t^3 Y_t^3 = C(t, Y_t^2, Y_t^3; 0), \quad \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0.$$

The last equality is a special case of the **balance condition**. It ensures that the wealth of a replicating portfolio falls to 0 at default time.

$dY_t = Y_t r dt$, $dY_t^2 = Y_t^2(\mu_2 dt + \sigma_2 dW_t)$, $dY_t^3 = Y_{t-}^3(\mu_3 dt + \sigma_3 dW_t - dM_t)$

Three assets, no total default

In the case where

$$\begin{aligned} dY_t^1 &= rY_t^1 dt \\ dY_t^2 &= Y_t^2(\mu_2 dt + \sigma_2 dW_t) \\ dY_t^3 &= Y_{t-}^3(\mu_3 dt + \sigma_3 dW_t + \varphi_3 dM_t) \end{aligned}$$

and $\varphi_3 \neq -1$.

In case of constant (or deterministic coefficients) **the riskneutral intensity is equal to the historical one**.

The price of a **total default claim** $h(Y^2_T, Y^3_T, H_T) = \mathbb{1}_{T<\tau}\tilde{h}(Y^2_T, Y^3_T)$ is solution of

$$\begin{aligned} C(\cdot,0)\,(r+\gamma_t) &= C_t(\cdot,0) + ry_2\partial_2 C(\cdot,0) + (r-\varphi_3\lambda_t)y_3\partial_3 C(\cdot,0) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C(\cdot,0) \\ C(T,y_2,y_3,0) &= \tilde{h}(y_2,y_3)\,. \end{aligned}$$

Example Consider a survival claim of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T<\tau\}} g(Y_T^3).$$

Then the post-default pricing function $C^g(\cdot\,;1)$ vanishes identically, and the pre-default pricing function $C^g(\cdot\,;0)$ solves

$$\begin{aligned} \partial_t C^g(\cdot\,;0) &+ r y_2 \partial_2 C^g(\cdot\,;0) + y_3 \left(r - \varphi_3 \gamma\right) \partial_3 C^g(\cdot\,;0) \\ &+ \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C^g(\cdot\,;0) - (r+\gamma) C^g(\cdot\,;0) = 0 \\ C^g(T, y_2, y_3; 0) &= g(y_3) \end{aligned}$$

Denote $\alpha = r - \varphi_3 \gamma$ and $\beta = \gamma(1 + \varphi_3)$.

Then, $C^g(t, y_2, y_3; 0) = e^{\beta(T-t)} C^{\alpha,g,3}(t, y_3)$ where $C^{\alpha,g,3}(t, y)$ is the price of the contingent claim $g(Y_T)$ in the Black-Scholes framework with the interest rate α and the volatility parameter equal to σ_3.

Let C_t be the current value of the contingent claim Y, so that

$$C_t = \mathbb{1}_{\{t<\tau\}} e^{\beta(T-t)} C^{\alpha,g,3}(t, y_3).$$

The hedging strategy of the survival claim is, on the event $\{t < \tau\}$,

$$\begin{aligned}
\phi_t^3 Y_t^3 &= -\frac{1}{\varphi_3} e^{-\beta(T-t)} C^{\alpha,g,3}(t, Y_t^3) = -\frac{1}{\varphi_3} C_t, \\
\phi_t^2 Y_t^2 &= \frac{\sigma_3}{\sigma_2} \left(Y_t^3 e^{-\beta(T-t)} \partial_y C^{\alpha,g,3}(t, Y_t^3) - \phi_t^3 Y_t^3 \right).
\end{aligned}$$

Hedging of a Recovery Payoff

The price C^g of the payoff $G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T\geq\tau\}} g(Y_T^2)$ solves

$$\begin{aligned}\partial_t C^g(\cdot\,;1) + ry\partial_y C^g(\cdot\,;1) + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy} C^g(\cdot\,;1) - rC^g(\cdot\,;1) &= 0\\ C^g(T,y;1) &= g(y)\end{aligned}$$

hence $C^g(t, y_2, y_3, 1) = C^{r,g,2}(t, y_2)$ is the price of $g(Y_T^2)$ in the model Y^1, Y^2. Prior to default, the price of the claim solves

$$\begin{aligned}\partial_t C^g(\cdot;0) &+ ry_2\partial_2 C^g(\cdot\,;0) + y_3\left(r - \varphi_3\gamma\right)\partial_3 C^g(\cdot\,;0)\\ &+ \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C^g(\cdot\,;0) - (r+\gamma)C^g(\cdot\,;0) = -\gamma C^g(t, y_2; 1)\\ C^g(T, y_2, y_3; 0) &= 0\end{aligned}$$

Hence $\quad C^g(t, y_2, y_3; 0) = (1 - e^{\gamma(t-T)})C^{r,g,2}(t, y_2).$

$dY_t = Y_t r dt,\ dY_t^2 = Y_t^2(\mu_2 dt + \sigma_2 dW_t),\ dY_t^3 = Y_{t-}^3(\mu_3 dt + \sigma_3 dW_t + \varphi_3 dM_t)$

Two defaultable assets with total default

Assume that Y^1 and Y^2 are defaultable tradeable assets with zero recovery and a common default time τ.

$$dY^i = Y^i(\mu_i dt + \sigma_i dW_t - dM_t), i = 1, 2$$

Then

$$Y_t^1 = 1\!\!1_{\{\tau > t\}} \widetilde{Y}_t^1, \quad Y_t^2 = 1\!\!1_{\{\tau > t\}} \widetilde{Y}_t^2$$

with

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i((\mu_i + \gamma_t)dt + \sigma_i dW_t), i = 1, 2$$

The wealth process V associated with the self-financing trading strategy (ϕ^1, ϕ^2) satisfies for $t \in [0, T]$

$$V_t = Y_t^1 \left(V_0^1 + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^{2,1} \right)$$

where $\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2 / \widetilde{Y}_t^1$.

Obviously, this market is **incomplete, however, some contingent claims are hedgeable**, as we present now.

Hedging Survival claim: martingale approach

A strategy (ϕ^1, ϕ^2) replicates a survival claim $C = X \mathbb{1}_{\{\tau > T\}}$ whenever we have

$$\widetilde{Y}^1_T \left(\widetilde{V}^1_0 + \int_0^T \phi^2_t \, d\widetilde{Y}^{2,1}_t \right) = X$$

for some constant $\widetilde{V}^1_0$ and some $\mathbf{F}$-predictable process ϕ^2.

It follows that if $\sigma_1 \neq \sigma_2$, **any survival claim $C = X \mathbb{1}_{\{\tau > T\}}$ is attainable.**

Let $\widetilde{Q}$ be a probability measure such that $\widetilde{Y}^{2,1}_t$ is an $\mathbf{F}$-martingale under $\widetilde{Q}$. Then the pre-default value $\widetilde{U}_t(C)$ at time t of $(X, 0, \tau)$ equals

$$\widetilde{U}_t(C) = \widetilde{Y}^1_t \, E_{\widetilde{Q}} \left(X (\widetilde{Y}^1_T)^{-1} \,|\, \mathcal{F}_t \right).$$

Example: Call option on a defaultable asset We assume that $Y_t^1 = D(t,T)$ represents a defaultable ZC-bond with zero recovery, and $Y_t^2 = 1\!\!1_{\{t<\tau\}}\widetilde{Y}_t^2$ is a generic defaultable asset with total default. The payoff of a call option written on the defaultable asset Y^2 equals

$$C_T = (Y_T^2 - K)^+ = 1\!\!1_{\{T<\tau\}}(\widetilde{Y}_T^2 - K)^+$$

The replication of the option reduces to finding a constant x and an **F**-predictable process ϕ^2 that satisfy

$$x + \int_0^T \phi_t^2 \, d\widetilde{Y}_t^{2,1} = (\widetilde{Y}_T^2 - K)^+.$$

Assume that the volatility $\sigma_{1,t} - \sigma_{2,t}$ of $\widetilde{Y}^{2,1}$ is deterministic. Let $\widetilde{F}_2(t,T) = \widetilde{Y}_t^2(\widetilde{D}(t,T))^{-1}$

$dY_t^i = Y_{t-}^i(\mu_i dt + \sigma_i dW_t - dM_t), i = 1, 2$

The credit-risk-adjusted forward price of the option written on Y^2 equals

$$\widetilde{C}_t = \widetilde{Y}_t^2 \mathcal{N}\big(d_+(\widetilde{F}_2(t,T),t,T)\big) - K\widetilde{D}(t,T)\mathcal{N}\big(d_-(\widetilde{F}_2(t,T),t,T)\big),$$

where

$$d_\pm(\widetilde{f},t,T) = \frac{\ln \widetilde{f} - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_{1,u} - \sigma_{2,u})^2 \, du.$$

Moreover the replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi_t^1 = -K\mathcal{N}\big(d_-(\widetilde{F}_2(t,T),t,T)\big), \quad \phi_t^2 = \mathcal{N}\big(d_+(\widetilde{F}_2(t,T),t,T)\big).$$

Begin at the beginning, and go on till you come to the end. ***Then, stop***.

L. Carroll, Alice's Adventures in Wonderland

C'est la fin du livre, mais ce n'est pas la fin de la recherche.

This is the end of the book, but not the end of the research.

Saint Augustin.