Valuation of Synthetic CDOs

Ben De Prisco, Alexander Kreinin
Algorithmics Inc.

RiskLab International Conference
Madrid, October 2005
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Publications:
Ian Iscoe, Alex Kreinin: “Pricing and Valuation of Synthetic CDOs “ (Journal of
Computational Finance)
Ian Iscoe, Alex Kreinin: “Pricing and Valuation of Basket Default Swaps“
(Journal of Computational Finance)
Ben De Prisco, Ian Iscoe, Alex Kreinin: “Loss in Translation “ (Risk Magazine)
Outline:

• Financial introduction into CDOs.
• Monte Carlo Pricing.
• Balance Equations.
• Compound Poisson approximation.
• Recursive Methods
• CDO Valuation Methods: Performance & Accuracy.
• Future Directions
Basket Credit Derivatives

1. Application of Limit Theorems of Probability Theory if the number of names is large.
2. Efficient recursion for basket default event distribution (when the number of names is small).
3. Optimized quadratures designed for CD payoff functions
Simplest form of synthetic credit derivative: single name credit default swap

Periodic payments

Protection buyer → Protection seller

Payment triggered by credit event

Basket default swap

The default model is more complicated

Credit event triggering payment may be 1\textsuperscript{st}, 2\textsuperscript{nd}, 3\textsuperscript{rd}, default => two elements of randomness

Two things are important: who and when
True CDO

Collateral Pool of Credit Risky Instruments:
• Bonds
• Loans
• CDS
• ABS

Tranched Securities:
• Equity
• Mezzanines
• Senior

Cash flow distribution policy: “waterfall” diagram: Super-Senior, Senior, Mezzanine tranches get first money flow and only in the end Equity tranche holders get leftovers.

**Collateral Pool is actively managed.**

Pool needs to satisfy various quality tests. (Overcollateralization, interest coverage, diversity etc…)

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Synthetic CDO

Synthetic Pool of Credit Risky Instruments:
• CLNs
• CDS

Tranched Securities:
• Equity
• Mezzanines
• Senior

Premiums and Default payout cashflows accrue to various tranches in pre-specified allocations.

**Synthetic Pool is NOT actively managed (i.e. underlying names do not change).**

Synthetic CDOs often referred to as Tranched Portfolio Default Swaps
CDO Pool Cash Flow

20 names

$2M = 100,000 \times 20$

$10M = 2M + 2M + 2M + 2M$

2 def. + 1 def. + 3 def. + 3 def.

2 + 1.8 + 1.7 + 1.4 + 1.1 = 8

T0 T1 T2 T3 T4
Synthetic CDO Tranches

Cumulative losses

2 def. 1 def. 3 def. 3 def.

T₃  T₂  T₁  T₀

0.2 0.5 1.1 2

Super-senior

Mezzanine

Equity

T₁  T₂  T₃

20.0 20.0 20.0

12.5 12.5 12.5

2.7 2.7 2.7

0.8 0.8 0.8

0.3 0.3 0.3

0.3 0.3 0.3
Tranche Losses = min\{S, \text{max}(L_i-1,0)\}
Balance Equation for CDO

\[
E \left[ \sum_{i=1}^{n} s_i (t_i - t_{i-1}) \cdot (S^{(j)} - L_i^{(j)}) \cdot d_i \right] = E \left[ \sum_{i=1}^{n} (L_i^{(j)} - L_{i-1}^{(j)}) \cdot d_i \right]
\]

\[
S = \frac{\sum_{i=1}^{n} (EL_i - EL_{i-1}) \cdot d_i}{\sum_{i=1}^{n} (t_i - t_{i-1}) \cdot (S - EL_i) \cdot d_i}
\]

The problem is reduced to computation of EL_i
MC Pricing: General Idea

*Expectation of premium payments* = *Expected Losses*

1. Simulate correlated default events through time (correlated default times) for names in underlying pool

2. Under each path, apply the pay-offs of either N\textsuperscript{th} to Default basket or CDO tranche

3. Value represented by average value across all MC paths
Conditional Independence

$(X, \mu)$ – generic space of (latent) variables with a measure $\mu$;

\[
X = x: \quad P(x) = \begin{pmatrix}
    p^{(1)}_1(x) & p^{(1)}_2(x) & \ldots & p^{(1)}_n(x) \\
p^{(2)}_1(x) & p^{(2)}_2(x) & \ldots & p^{(2)}_n(x) \\
    \vdots & \vdots & \ddots & \vdots \\
p^{(K)}_1(x) & p^{(K)}_2(x) & \ldots & p^{(K)}_n(x)
\end{pmatrix}
\]

Let $N = (N^{(1)}, \ldots, N^{(K)})$ be vector of recovery-adjusted notionals

\[
EL_i(x) = F(P(x), N, l, u)
\]

\[
EL_i = \int EL_i(x) \, d\mu(x) = \int F(P(x), N, l, u) \, d\mu(x)
\]
The Space X and Measure $\mu$

$X - C([0, T]), \mu - Gaussian$ :

Iscoe, Kreinin, Rosen – ARQ (1999)

$X = R^m, \mu - Gaussian, T$-distribution:

De Prisco, Iscoe, Kreinin (2005)) Risk : Compound Poisson Approximation of the pool loss distribution
Correlation Model

Correlated Credit Events for Estimation of Losses

Merton’s Model

$Y_t^{(k)} \equiv Y^{(k)} = \sum_{m=1}^{M} \alpha_m^{(k)} X_m + \beta^{(k)} \epsilon^{(k)},$

How can we measure creditworthiness?

Systemic Component

Specific Component
Conditional independence framework

\[ Y_{(k)}^{(i)} = Y_{(k)} = \sum_{c=1}^{C} \alpha_{c}^{(k)} X_{c} + \beta^{(k)} \epsilon^{(k)}, \quad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, K, \]

CWI  Credit drivers  Idiosyncratic component

\[ \sum_{c=1}^{C} \sum_{d=1}^{C} \alpha_{c}^{(k)} R_{cd} \alpha_{d}^{(k)} + (\beta^{(k)})^2 = 1. \]

Conditional default probabilities

\[ \hat{\pi}_{(k)}^{(i)}(x) = \Phi \left( \left[ H_{(k)}^{(i)} - \sum_{c=1}^{C} \alpha_{c}^{(k)} x_{c} \right] / \beta^{(k)} \right). \]

Unconditional default probability

\[ p = \Phi(H) \]
Conditional default probability

\[ \hat{\pi}_x^{(k)}(x) = \Phi \left( \frac{[H_x^{(k)} - \sum_{\alpha=1}^{C} \alpha x^{(k)} \beta^{(k)}}}{\beta^{(k)}} \right) \]
Unconditional probabilities

\[ \Pi^{(k)}_{\hat{\tau}} = \int_{\mathbb{R}^n} \Pi^{(k)}_{\hat{\tau}}(x) \, d\rho(x), \quad k = 1, 2, \ldots, K, \quad i = 1, 2, \ldots, n, \]

Unconditional mean discounted losses

\[ \mathbb{E} \left( L_{\tau} d_\tau \right) = \int_{\mathbb{R}^n} \mathbb{E}_x \left( L_{\tau} d_\tau \right) \, d\rho(x), \]

\[
(m - 1) \cdot \Pr \left( \tau_m = \tau^{(k)}; \tau_m \in (t_{i-1}, t_i] \right) = \sum_{j \neq k} \Pr \left( \tau_{m-1} (B^j) = \tau^{(k)}; \tau_{m-1} \in (t_{i-1}, t_i] \right) \\
- (K + 1 - m) \cdot \Pr \left( \tau_{m-1} = \tau^{(k)}; \tau_{m-1} \in (t_{i-1}, t_i] \right)
\]

\[
\Pr \left( \tau_m = \tau^{(k)}; \tau_m \in (t_{i-1}, t_i] \right) = \Pi_i^{(k)}(m) \\
\Pr \left( \tau_{m-1} = \tau^{(k)}; \tau_{m-1} \in (t_{i-1}, t_i] \right) = \Pi_i^{(k)}(m - 1)
\]
Calibration of Default Boundary

Risk Neutral Probability of Default Term Structure for each name (calibrated from Par CDS quoted spreads $\Rightarrow$ hazard rates)

$$\mathbb{P}\left(T^{(k)} \leq t_i\right) = \hat{\pi}_i^{(k)}, \quad i = 1, 2, \ldots, n.$$

Translated to Default Boundary Curve for each name

$$H_i^{(k)} = \Phi^{-1}(\hat{\pi}_i^{(k)}),$$
Name creditworthiness $Y^{(k)}$ sampled under each MC run

$$\tau^{(k)} = \min\{t_i : Y^{(k)}(t_i) < H_i^{(k)}\}$$

$$(\tau^{(k)} = +\infty \text{ if } Y^{(k)}(t_i) \geq H_i^{(k)} \text{, for all } 1 \leq i \leq n).$$

Sample 1: $Y^{(k)} = -1.5$
Sample 2: $Y^{(k)} = -2.1$
Requires 1000s of pricing paths under single scenario to estimate value appropriately!

Need fast analytic approximations.
H-W approach, as well as various other analytic flavours generally require the derivation of entire pool loss distribution before tranche profiles are overlaid.

This is a time consuming process and ignores the fact that investors may often own only certain tranches.

To speed up computation time, pool loss distribution should focus on relevant bands for the tranches held.
Tranche Losses

Tranche Structure

$S^{(j)}$

Tranche Losses

$S^{(1)}$

Equity Tranche losses

$\begin{align*}
L^{(j)} &= (L - l(j))^+ - (L - l(j) - S(j))^+
\end{align*}$

Tranche losses

Pool losses (L)

Pool losses (L)
Therefore, we only need to integrate the distribution for the bands we care about!!!
Consider the simplest case: instruments in pool have common notional and common term structure of default probabilities:

\[ \pi_i = P(\tau = t_i), \quad i = 1, 2, \ldots, n \]

Conditional default probabilities:

\[ p_i = P(\tau = t_i | \tau \geq t_i) \]

\[ p_i = \frac{\pi_i}{1 - \sum_{m=1}^{i-1} \pi_m} \]

\[ \pi_i = p_i \cdot \prod_{m=1}^{i-1} q_m \]

Therefore, the number of instruments in default at time \( t_i \), has Binomial distribution:

\[ P(\nu_1 = k_1) = \binom{K}{k_1} p_1^{k_1} q_1^{K-k_1}, \quad q_1 = 1 - p_1, \]
Therefore, for a given tranche, expected losses can be derived directly as:

\[
\mathbb{E} \left[ L^{(j)}_i \right] = \sum_{1 \leq k < \frac{l^{(j)}}{N_1}} (N_1 k - l^{(j)}) \text{Bin}(k; K, \hat{\pi}_i) + S^{(j)} \sum_{\frac{l^{(j+1)}}{N_1} \leq k \leq K} \text{Bin}(k; K, \hat{\pi}_i). \quad (3.15)
\]

where \( \hat{\pi}_i = \sum_{m=1}^{i} \pi_m, 1 \leq i \leq n \) represents cumulative default probabilities.
Convergence to Poisson process

Independent default events

\[ L_i = N^{(1)} \cdot \nu_i, \] where \( \nu_i \) is the number of default events by the time \( t_i \).

\[ \nu_i = \sum_{k=1}^{K} I(\tau^{(k)} \leq t_i); \quad \nu_i \Rightarrow \text{Pois}(\hat{\pi}_i) \]

\[ \hat{\pi}_i = \sum_{m=1}^{i} \pi_m, \quad 1 \leq i \leq n \]

\[ \nu_i(x, K) \Rightarrow \text{Pois}(\hat{\pi}_i(x)) \quad \text{(as } K \to \infty) \]

\[ \nu_i(K) \Rightarrow \nu_i^* \]

\[ \nu(K) \Rightarrow \nu^* \]
Poisson approximation for homogeneous pool

If the pool is large enough, then $v_i$ is approximated by $\text{Poisson}(\hat{\lambda}_i)$.

$$P(v_i = k) = e^{-\hat{\lambda}_i} (\hat{\lambda}_i)^k / k!, \quad k = 0, 1, \ldots$$

Therefore expected tranche losses can be calculated as:

$$E[L_i^{(j)}] \approx \sum_{l(j) \leq k < l(j+1)} (N_i k - l^{(j)}) \text{Pois}(k; \hat{\lambda}_i) + S^{(j)} \sum_{l(j+1) \leq k < K} \text{Pois}(k; \hat{\lambda}_i)$$

where $\hat{\lambda}_i = K \hat{\pi}_i, 1 \leq i \leq n$.
A compound Poisson process can be represented as:

\[ X(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0 \]

Where \( \{N(t), t \geq 0\} \) is a Poisson process, and \( \{Y_n, n \geq 0\} \) is a family of independent and identically distributed random variables which are also independent of \( \{N(t), t \geq 0\} \).
Heterogeneous Pool

Assume all underlying instruments in the pool have different notionals and individual term structures of default probabilities:

\[ \pi_i^{(k)} = P(\tau^{(k)} = t_i), \quad i = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, K; \]

Generally speaking:

\[ \hat{\pi}_i^{(k)} = \sum_{m=1}^{i} \pi_m^{(k)}, \quad 1 \leq i \leq n \]

\[ \hat{\lambda}_i = \sum_{k=1}^{K} \hat{\pi}_i^{(k)}, \quad 1 \leq i \leq n \]

\[ f_i(N) = \sum_{k:N_k=N} \hat{\pi}_i^{(k)}/\hat{\lambda}_i, \quad 1 \leq i \leq n, \quad N = 1, 2, \ldots \]

Note: \( f_i \) is simply the relative frequency of notional values
Consider a random variable for Notionals in the pool denoted by $N_i$ as having the probability mass function $f_i$ described previously, then the distribution of pool losses can be represented approximately by the following compound Poisson process:

$$L_i \overset{D}{=} \sum_{m=1}^{N_i} N_i^{(m)}$$

where $(N_i^{(m)})_{m=1}^K$ is an i.i.d. sequence of copies of $N_i$ and independent of $M_i \sim \text{Pois}(\hat{\lambda}_i)$

**Interpretation:**

Pool default events arrive in accordance with a Poisson process. The loss suffered with each default event is assumed to be independent and identically distributed. Then $\{L_i, t \geq 0\}$ is a **compound Poisson process** where $L_i$ denotes the cumulative default losses incurred by $t$ and $N_i^{(m)}$ represents the loss of the $m^{th}$ default.
Therefore expected tranche losses can be calculated as:

\[
\mathbb{E}L_i^{(j)} = S^{(j)}(1 - e^{-\lambda_i}) - e^{-\lambda_i} \left\{ S^{(j)} \sum_{m=1}^{t^{(j)}} \frac{\hat{\lambda}_i^m}{m!} \sum_{m \leq N \leq t^{(j)}} f_i^{*m}(N) \right. \\
+ \sum_{m=1}^{t^{(j+1)} - 1} \frac{\hat{\lambda}_i^m}{m!} \sum_{t^{(j)} < N < t^{(j+1)}} \left[ t^{(j+1)} - N \right] f_i^{*m}(N) \left. \right\}
\]
Heterogeneous Pool

1. Scenarios (integration pts)

2. Conditional probabilities \( p(X) \)

3. Obligor losses \( l \)

4. Tranche Loss Profiles

4a. Tranche Expected Losses

5. Unconditional Expected Tranche Losses

\[ p_j(X=x_1) \quad j=1,\ldots,n \]
\[ p_j(X=x_2) \quad j=1,\ldots,n \]
\[ p_j(X=x_3) \quad j=1,\ldots,n \]
How many MC pricing paths are required?

Assume: Homogeneous pool where $K=100$ and initial credit ratings same

$T=5$yrs, Equity 3%, Mezz. Jr. 1%, Mezz. 2.1%, Senior 6%, S-Senior 87.9%

### MC Error Estimates

95% conf intervals based on 10,000 paths:

95% conf intervals based on 100,000 paths:

**Conclusion:** Reasonable error bounds with 100,000 paths
Accuracy of Poisson Approximation
For homogeneous portfolio, binomial represents “true” value

<table>
<thead>
<tr>
<th>K</th>
<th>Tranche</th>
<th>Baa2 Spreads (bp)</th>
<th></th>
<th>Baa1 Spreads (bp)</th>
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<tr>
<td></td>
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<td>Poisson</td>
<td>Binomial</td>
<td>Poisson</td>
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<td>200</td>
<td>Equity</td>
<td>978</td>
<td>978</td>
<td>246</td>
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<td></td>
<td>Mezzanine Jr.</td>
<td>7</td>
<td>6</td>
<td>0</td>
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<tr>
<td></td>
<td>Mezzanine</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Senior</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>150</td>
<td>Equity</td>
<td>977</td>
<td>977</td>
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<td>7</td>
<td>6</td>
<td>0</td>
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<td>Mezzanine</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td></td>
<td>Senior</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
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<td>Equity</td>
<td>958</td>
<td>958</td>
<td>246</td>
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<td>43</td>
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<td>4</td>
<td>3</td>
<td>0</td>
</tr>
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<td></td>
<td>Senior</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>75</td>
<td>Equity</td>
<td>938</td>
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<td>246</td>
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<td>69</td>
<td>1</td>
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<td>9</td>
<td>0</td>
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<td>Senior</td>
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<td>0</td>
<td>0</td>
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<tr>
<td>50</td>
<td>Equity</td>
<td>898</td>
<td>898</td>
<td>244</td>
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<td>Mezzanine Jr.</td>
<td>118</td>
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<tr>
<td></td>
<td>Senior</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: CDO spreads: Poisson approximation vs. Binomial analytics

Conclusion:
Poisson is reasonable approximation for pools even as small as 50 names.
Accuracy of Poisson Approximation

Heterogeneous pool

Poisson comparison vs 100,000 MC paths

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Poisson</th>
<th>MC</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>736</td>
<td>740</td>
<td>[730 750]</td>
</tr>
<tr>
<td>Mezzanine Jr.</td>
<td>392</td>
<td>394</td>
<td>[387 401]</td>
</tr>
<tr>
<td>Mezzanine</td>
<td>285</td>
<td>285</td>
<td>[281 290]</td>
</tr>
<tr>
<td>Senior</td>
<td>110</td>
<td>110</td>
<td>[105 112]</td>
</tr>
<tr>
<td>Super-Senior</td>
<td>3</td>
<td>3</td>
<td>[3 3]</td>
</tr>
</tbody>
</table>

Table 8: Tranche spreads (bp): MC vs. Poisson approximation.

Poisson results always contained within 95% MC conf intervals based on 1

Conclusion: Poisson approximation is as reliable as MC pricing with 100,000 paths.
Quadratures

Tranche Losses

\[ L^{(j)} = (L - l(j))^+ - (L - l(j) - S(j))^+ \]
1) **HW recursive approach** is faster than HW bucketed convolution approach. This approach was implemented along with the following “enhancements”:

- Modified recursion formula to remove instability problems with the suggested approach
- Gauss-Hermite integration points (for credit driver) replaced with Willow points thus only requiring 14 points for integration (vs upwards of 30)
- Pool loss distributions only calculated once and re-used for several tranches using same pool

2) **Compound Poisson approximation**. This approach will be faster than approach above in cases where:

- Only 1 or 2 tranches are owned (accounting for small % of entire pool)
- The dispersion of assumed recovery rates in the pool are low

3) **Standard Monte Carlo pricing** – for those who have time to kill (and does not trust analytics)
CDO Performance

Assuming the number of names in the pool is fixed, performance will depend on 3 dimensions:

- Number of homogeneous pools
- Dispersion of recovery rates

**Performance Numbers for CDOs**

100 Names in the Pool.
100 single step scenarios.
Monte Carlo pricing (100k paths) takes 585s.
All times measured in seconds.
Times will scale linearly by # of scenarios and time steps.

<table>
<thead>
<tr>
<th></th>
<th>1 Tranche (6%)</th>
<th>4 Tranches (12.125%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High Dispersion</td>
<td>Low Dispersion</td>
</tr>
<tr>
<td></td>
<td>EHW</td>
<td>Algo</td>
</tr>
<tr>
<td>1 Pool (Homogeneous)</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>2 Pools</td>
<td>10.0</td>
<td>16.2</td>
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<tr>
<td>3 Pools</td>
<td>10.8</td>
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<tr>
<td>4 Pools</td>
<td>20.3</td>
<td>25.2</td>
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<tr>
<td>5 Pools</td>
<td>23.7</td>
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<tr>
<td>6 Pools</td>
<td>25.9</td>
<td>30.3</td>
</tr>
<tr>
<td>7 Pools</td>
<td>28.6</td>
<td>35.5</td>
</tr>
</tbody>
</table>

Summary:

- Compound Poisson process will be faster in cases of single tranches with low recovery rate dispersion.
- EHW/Algo algorithms will be anywhere between 20x-100x faster than MC pricing!!!!!!
Heterogeneous Pool

Query:

What if someone does own all the tranches in the pool. Is there still a benefit to the Algo approach?

Response:

YES! Due to the fact that:

$$E(PoolLosses)_i = \sum_{j=1}^{m} E(L^{(j)}_i)$$

we never need to integrate for the entire pool distribution. We can calculate the expected tranche losses for Equity, B and C and back out A’s expected tranche losses through the relationship above.

In fact, if we ever face a tranche that covers more than half the pool, it makes sense to apply the Algo Approach to the complementary tranche and back out the desired expected tranche losses since the complement will be faster to
1. Dynamic correlation model
2. Robust calibration algorithm
3. CDO of CDOs (valuation, calibration, sensitivities)
Part I
Loss in Translation
The End