A BAYESIAN SOLUTION TO THE EQUITY PREMIUM PUZZLE

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Statistical Laboratory, Cambridge

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leads to equilibrium prices

$$S_t = E_t \left[\sum_{j\geq 0} \beta^j \frac{U'(y_{t+j})}{U'(y_t)} \, y_{t+j} \right], \qquad B_t = \beta E_t \left[\frac{U'(y_{t+1})}{U'(y_t)} \right].$$

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- bond price is constant!
- stock price is only finite if $\beta E[\, \exp\{(1-R)\xi_1\} \,] < 1.$

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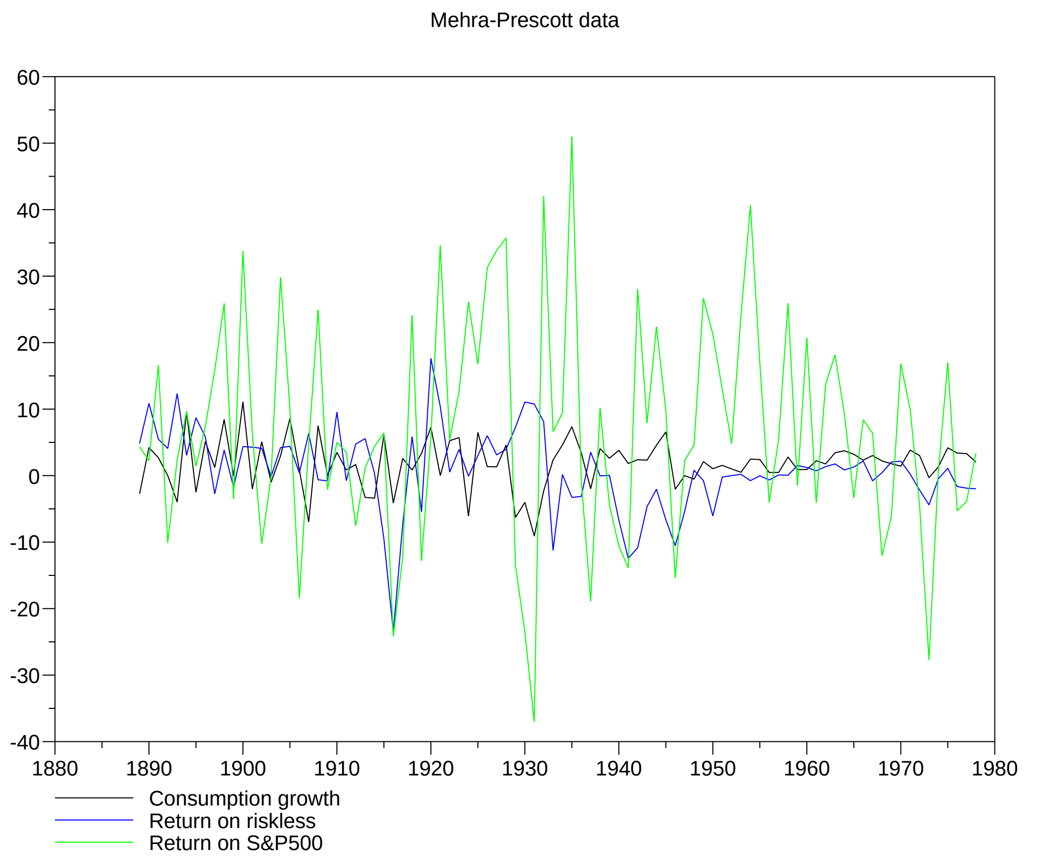

Mehra-Prescott ctd ...

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Kocherlakota (JEL 96) and Mehra & Prescott (Handbook of the Economics of Finance 2003) can't endorse any of these alternatives ...

The 20's example

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Conclusion: We must treat μ as unknown, and infer it from the observed prices.

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For now, think $\varphi \equiv 1$, so we have a Gamma-Gaussian prior.

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$$\begin{aligned} \frac{\pi_t(\mu, \tau | \xi_{[1,t]})}{\varphi(\mu, \tau)} &\propto \tau^{\alpha_0 - 1}e^{-b_0\tau}e^{-\frac{1}{2}K_0\tau\mu^2}\sqrt{\tau} \,.\, e^{-\frac{1}{2}\tau\sum_{k=1}^{t}(\xi_k - \mu)^2}\tau^{t/2} \\ &\propto \tau^{\alpha_t - 1}e^{-b_t\tau}e^{-\frac{1}{2}K_t\tau(\mu - m_t)^2}\sqrt{\tau} \\ &\equiv \tilde{\pi}_t(\mu, \tau) \end{aligned}$$

where α_t, b_t, K_t depend on α_0, b_0, K_0 and $\xi_{[1,t]}$.

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where α_t, b_t, K_t depend on α_0, b_0, K_0 and $\xi_{[1,t]}$. Same form as before:

$$\tau \sim \Gamma(\alpha_t, b_t), \qquad \mu | \tau \sim N(m_t, (K_t \tau)^{-1}).$$

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So (writing $\nu = R - 1$, assumed positive (wmlog))

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$$S_t = \frac{y_t}{1 - \beta E(\exp\{-\nu\mu + \nu^2/2\tau\})}$$

which is finite iff $(\mu, \tau) \in A \equiv \{(\mu, \tau) : \log \beta - \nu\mu + \nu^2/2\tau < 0\}$.

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which is finite iff $(\mu, \tau) \in A \equiv \{(\mu, \tau) : \log \beta - \nu\mu + \nu^2/2\tau < 0\}$. For this reason, we choose

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We therefore have

$$S_t = y_t f_t(\xi_{[1,t]}\ ;\ \beta, R, \alpha_0, b_0, K_0), \qquad B_t = g_t(\xi_{[1,t]}\ ;\ \beta, R, \alpha_0, b_0, K_0).$$

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Econometrician sees

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- Introduces slight bias, but can always run the PF again at the end...

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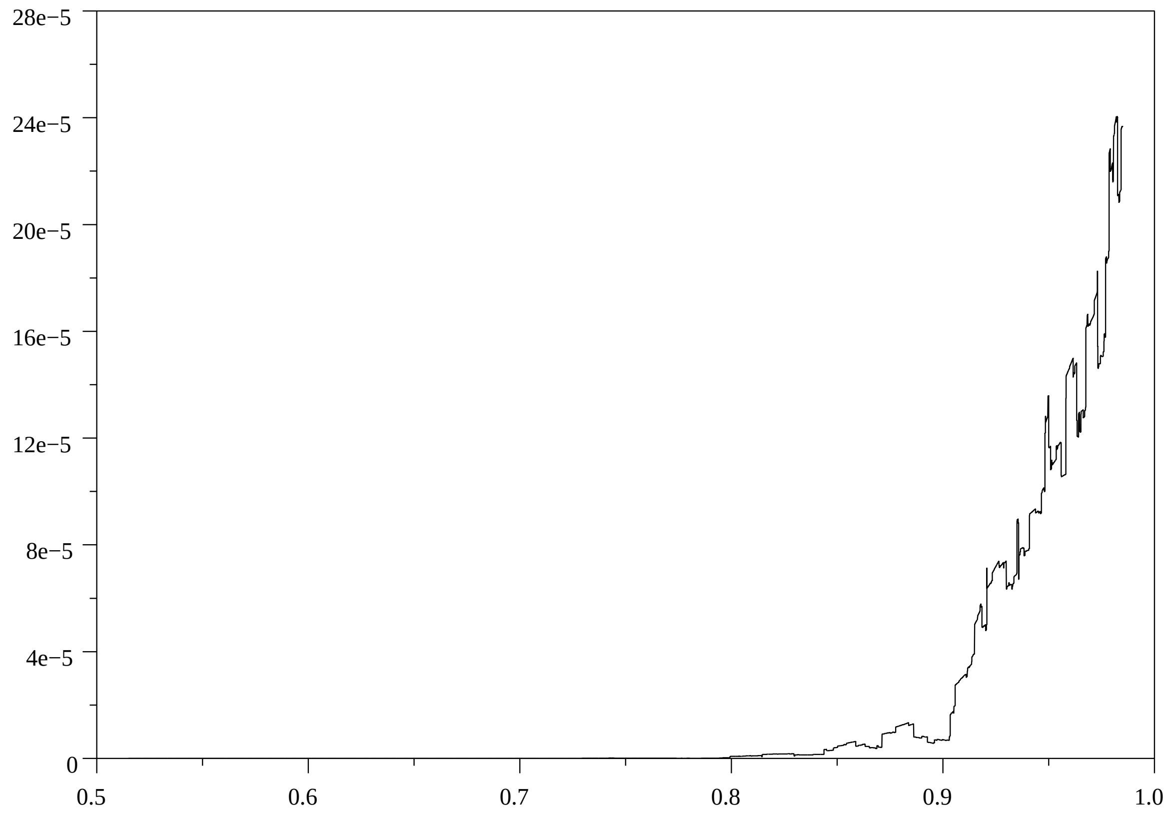

Figure 1: Posterior density for β

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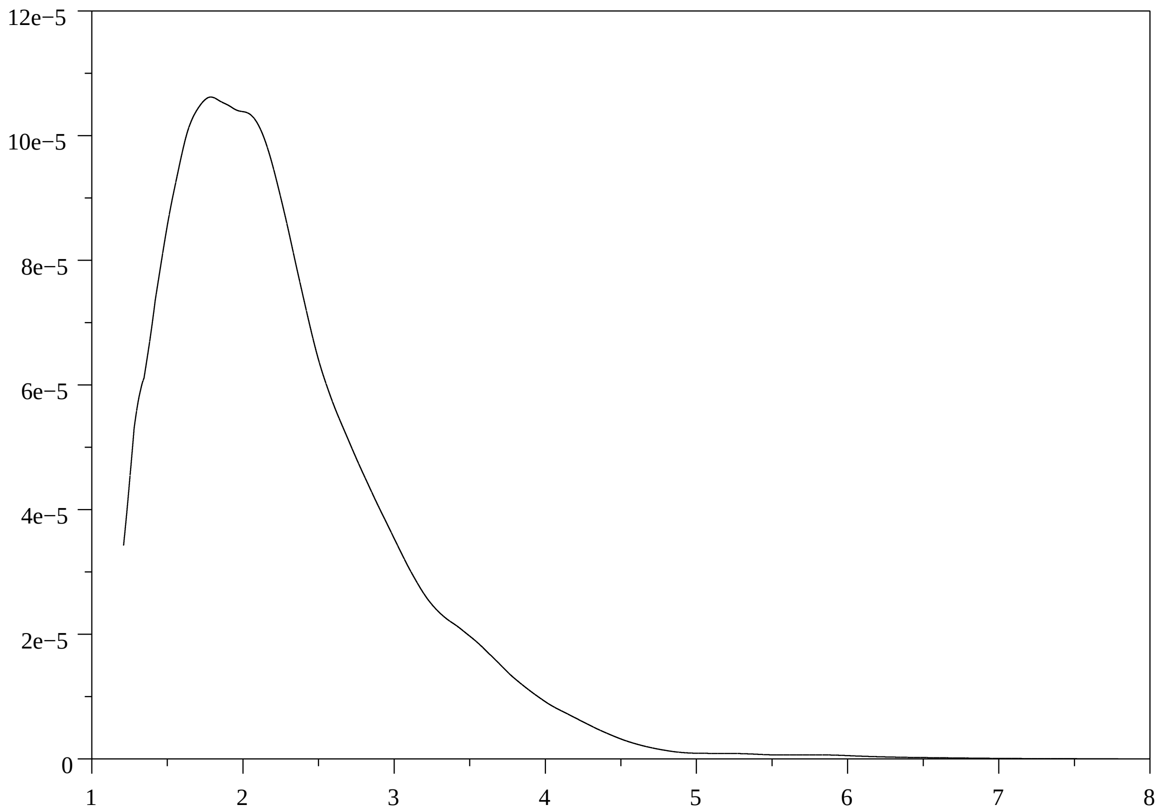

Figure 2: Posterior density for R

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.. but the apparently large value of the equity premium is not one of them, once you acknowledge and properly handle the enormous uncertainty in the rate of return of the stock index.