IIIª Jornada de Riesgos Financieros – RiskLab • Madrid

Madrid, 1 de Diciembre , 2003

Dynamic Portfolio Credit Risk : Credit Derivatives,Credit Spreads and Infections in Credit Portfolios

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Agenda

Dynamic Credit Risk Modelling

Observations of Dynamic Portfolio effects

Credit Spreads, Infections and Frailties

Evolutionary Portfolio Credit Management

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Dynamic Credit Risk Modelling

Realistic Risk Modelling

- Credit Risk Management needs a measure for :
  - Economic and regulatory capital calculations (Basel II A-IRB)
  - Loss reserving
  - Variability of losses
- The need to model key Risk measures such as :
  - Forward looking cycle average defaults
  - Dependence among defaults
  - Variation of the default probability over the credit cycle
  - Practitioners have to choose a functional form and estimate the parameters of the chosen model
  - Data limitations . Historical Data . Standard error around parameter estimates

THE CHOICE OF A REALISTIC MODEL IS A POSTULATION

Loss distribution

- Vasicek (1987,1997) . Inverse Gaussian distribution .
- Basically the values of the assets of an obligor are driven by one common factor Y and an idiosyncratic standard normal noise ε_i where :

$$V_i(T) = \sqrt{\rho} Y + \sqrt{1-\rho}\,\varepsilon_i$$

- The probability that the fraction L of defaults in the portfolio is less than a given level q is given by :

$$P[L \le q] = \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,\Phi^{-1}(q) - \Phi^{-1}(p)\right)\right)$$

- Where :
- p is the default probability of any obligor in the portfolio
- ρ is the asset value correlation between any two obligors .
- Φ is the cumulative standard normal distribution

- Normally the loss distribution will need very long numerical simulations
- Credit Metrics model can be seen as a more general model of which Vasicek is a limiting case (With $X_i := \Phi(V_i)$ and $p := \Phi(K)$)
- An interesting relationship is the conditional Probability of Default given the common factor Y(Also from Vasicek) :

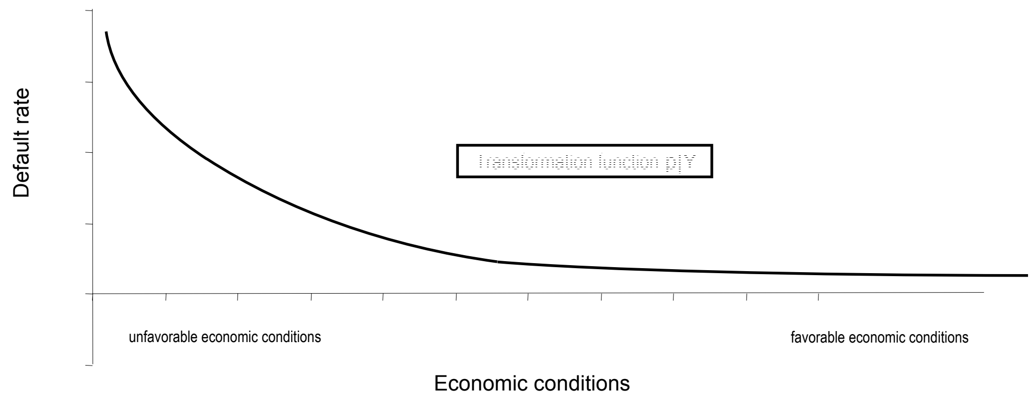
$$p|_Y = \Phi\left[\Phi^{-1}(\overline{p}) - \frac{\sqrt{\rho} Y}{\sqrt{1-\rho}}\right] = \Phi\left[c - \frac{\sqrt{\rho} Y}{\sqrt{1-\rho}}\right]$$

Where m is standard normal distributed

$\Phi[c] = \overline{p}$ with $\overline{p}$ as unconditional default probability

Loss distribution

Default rates and single factor approach Y



- Correlation is in the curvature of the transformation function (The conditional probability of default given Y) .
- Difficulties replicating the loss distribution for CDO's given that only ρ will be the variable to fit the main parts of the distribution : mezzanine,equity and senior tranches . Not enough degrees of freedom to match the loss distributions observed in the market
- Furthermore the shape of the distribution changes significantly when different time horizons (and thus default probabilites) are considered .
- We will not use then the multivariate normally distributed random variables to trigger default instead we will use Archimedean copula functions .

TRANSFORMATION FUNCTION CAN NOT PRICE GENERAL CDO DYNAMICS

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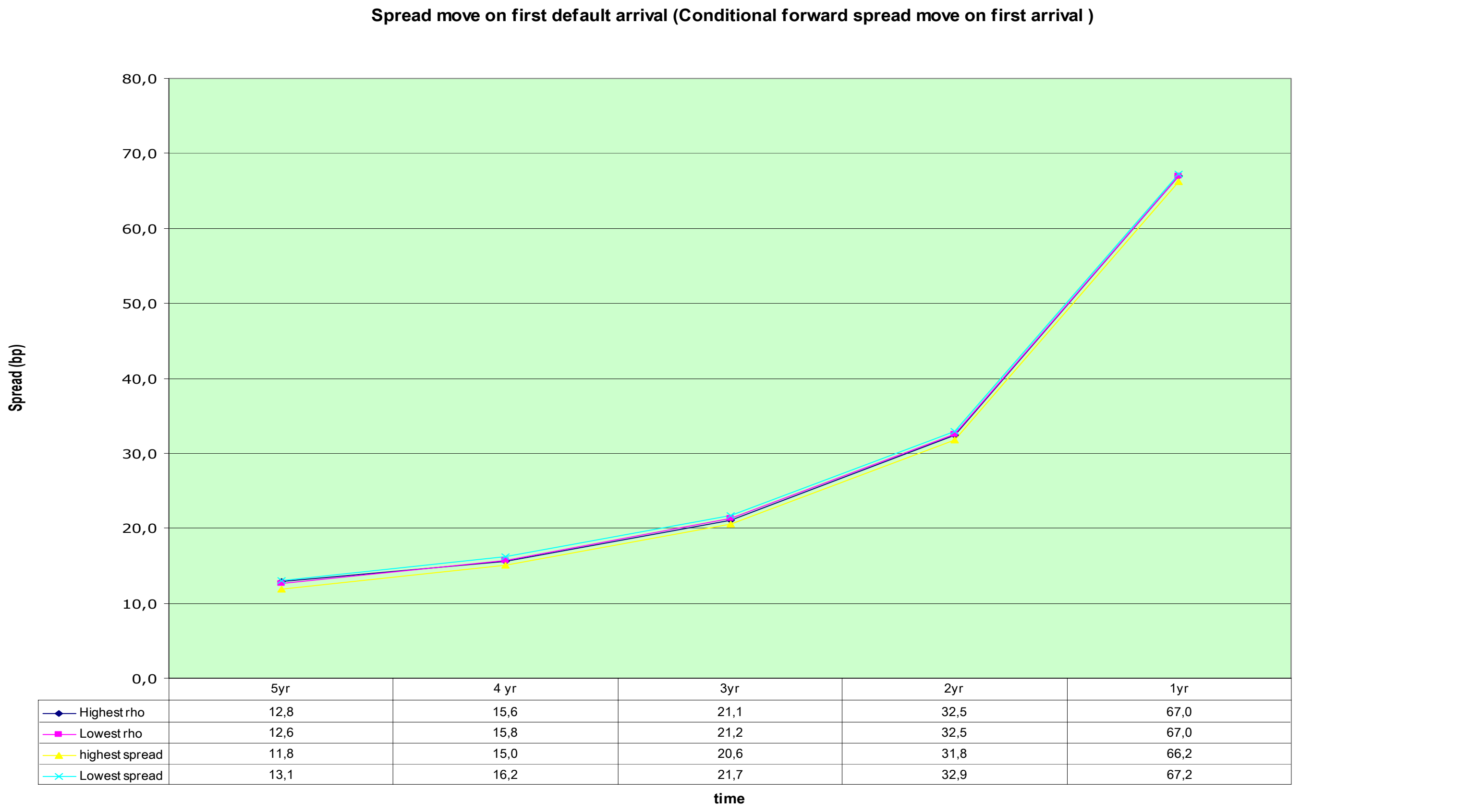
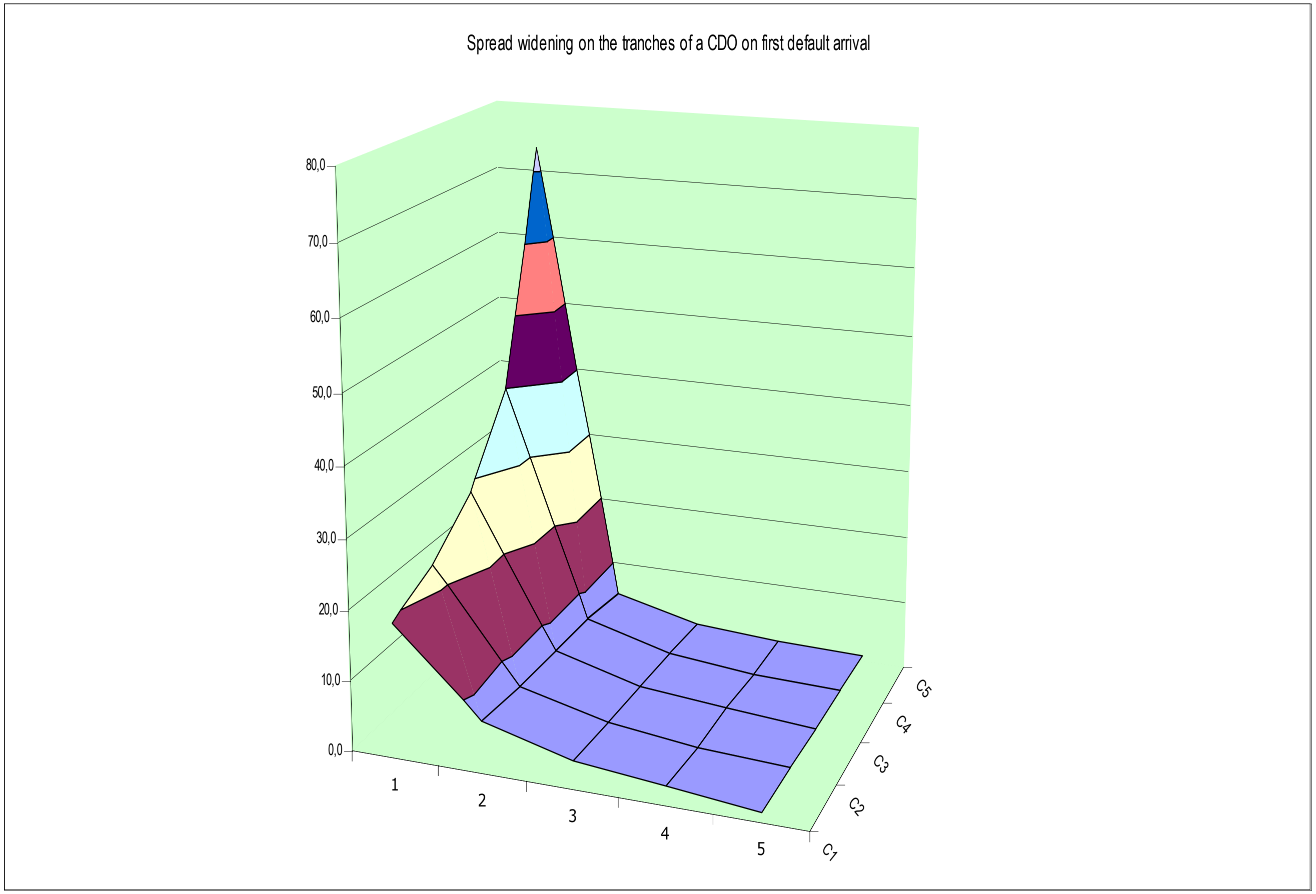
Observations of Dynamic Portfolio effects

Dynamic Portfolio effects

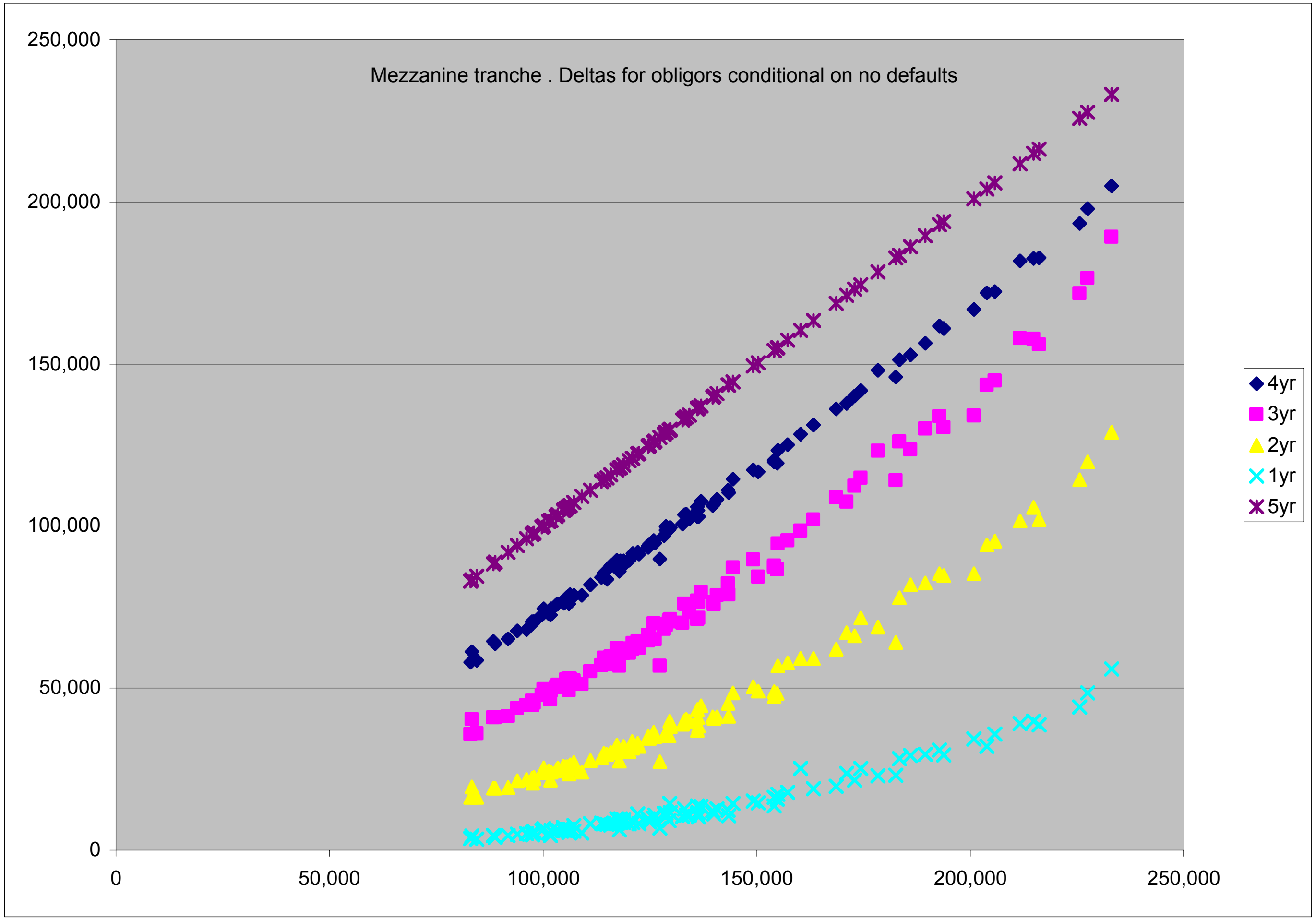


	5yr	4 yr	3yr	2yr	1yr
Highest rho	12,8	15,6	21,1	32,5	67,0
Lowest rho	12,6	15,8	21,2	32,5	67,0
highest spread	11,8	15,0	20,6	31,8	66,2
Lowest spread	13,1	16,2	21,7	32,9	67,2

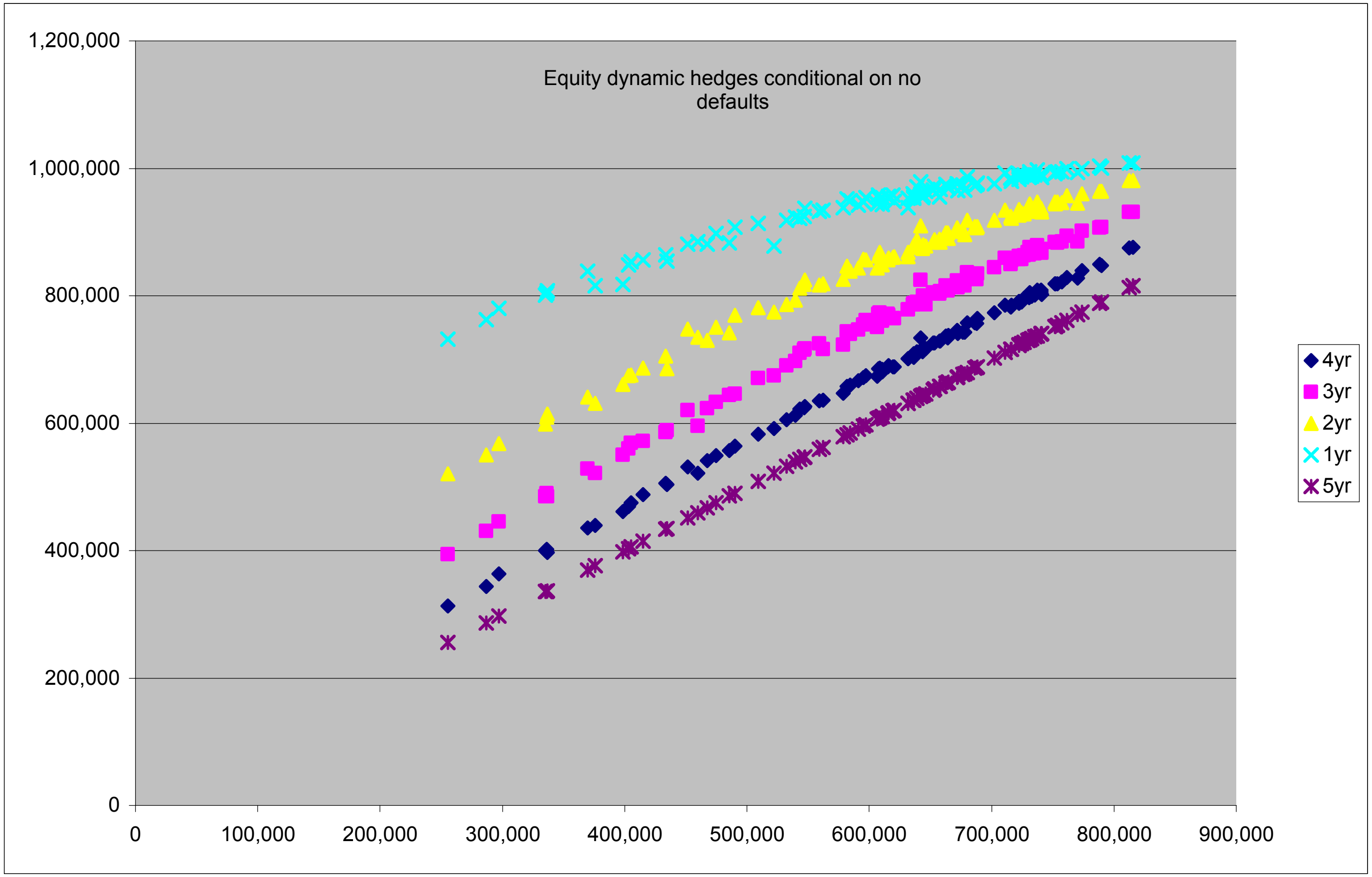
Dynamic Portfolio effects



Dynamic Portfolio effects



Dynamic Portfolio effects



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Credit Spreads, Infections and Frailties

The Frailty

- Two obligors i=1,2 . Both defaults will be triggered by a Poisson process of intensity h_i. Default times $\tau_{i\ ,\ i=1,2}$.
- The survival probabilities of the obligors are $P_i(t,T)=e^{-(T-t)h_i}$ if both Poisson are independent then the joint survival function is :

$$P(t,T_1,T_2) = P(\tau_1 > T_1, \tau_2 > T_2) = e^{-(T_1-t)h_1-(T_2-t)h_2} \qquad (1)$$

Let's assume that $\mathbf{h_1 = Y\,\lambda_1}$ and $\mathbf{h_2=Y\,\lambda_2}$,(2) , where Y is a continuous random variable with density $f_Y(y)$. Conditional to Y the default processes $N_1(t)$ and $N_2(t)$ are still independent processes with intensities h_1 and h_2 . **Y** is called a **frailty** in biostatistics (Hougaard 2000) . Normally the frailty of patients in a group and usually unobserved common influence . With initial value of the expectation of Y , E[Y]=1 .

We can interpret the frailty as the **uncertainty** around λ_i .

We could have chosen a more general case for the model in the econometric literature ie

$$h_i = e^{\left[\sum_{n=1}^{N} x_{in}\beta_n + \varepsilon_i\right]} = e^{\left[\sum_{n=1}^{N} x_{in}\beta_n\right]} e^{\left[\sum_{n=1}^{N} \varepsilon_i\right]} = Y_i e^{\left[\sum_{n=1}^{N} x_{in}\beta_n\right]} \qquad (3)$$

x_{in} With is n th covariate of the i th obligor

β_n Are coefficients of the influence of x_{in}

- **Initial Survival Probabilities**
- Using the same frailty for both we can say that they are independent if conditioned on the frailty Y . Therefore the model shows **conditional independence** .

Conditional on Y=y we have the initial joint survival function $P[\tau_1 > T_1 \wedge \tau_2 > T_2 \mid Y = y] = e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]}$ (5) with τ_i the time to default of obligor i

The Frailty

The joint conditional density function of the default times is

$$f(T_1, T_2 \mid y) = y^2 \lambda_1 \lambda_2 e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]} \qquad (6)$$

- Taking the expectation of the conditional joint probability we obtain the joint survival probability and the individual survival probabilities :

$$E[P[\tau_1 > T_1 \wedge \tau_2 > T_2]] = E[e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]}] \qquad P[\tau_i > T_i] = E[e^{-Y\lambda_i T_i}] \qquad (7)$$

- And the initial default hazard rate

$$h_i(0) = -\frac{\partial}{\partial T} E[e^{-Y\lambda_i T}]\,|_{T=0} = \lambda_i E[Y] \qquad (8)$$

- If the Laplace transform is known in close form :

$$L_Y(s) := E[e^{-sY}] \,, \qquad (9)$$

- Determines the distribution of Y and we can easily determine the expectations with respect to Y easily .

- Comparing equations 7 and 9 with 1 the introduction of the frailty is a replacement of the exponential function with the Laplace transform function . The exponential function is itself a Laplace function . If we choose Y to be $\Gamma(\alpha,\beta)$ then the Laplace transform is $(1+\beta s)^{-\alpha}$. (10)

The Frailty

- **Survival Probabilities at later times** .
- As time goes by and no defaults happen we gather information about the likely values of the frailty Y . We assume that we only have new information about observations of defaults or survival of the obligors . The result is that we should revise the distribution of Y if no defaults happen to making the values of the frailty lower . The contrary should happen too if we observe a default then there should be a high frailty we have under-estimated the value of Y .

- **Updating the frailty**
- Equation (6) gives us the joint conditional density of the default times and the frailty and (5) the conditional distribution . From this we will derive the conditional density of y , conditional on the observations that we have made on τ_1 and τ_2 .
- The survival of both until time t > 0 . By (5) and Y < y* has the probability $\int_0^{y} f_Y(y) e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]} dy$ and the unconditional probability of the survival event is $E[e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]}]$
- so by Bayes rule , the conditional probability of Y<y*

$$P[Y <= y^*] = \int_0^{y^*} f_Y(y) \frac{e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]}}{E[e^{[-Y(\lambda_1 T_1 + \lambda_2 T_2)]}]} dy \qquad (11)$$

- The conditional density of f_Y of Y conditional on the survival of both obligors is given by the exponential transformation of Y . The likelihood ratio of the transformation is

$$\frac{f_{Y,F_t}(y)}{f_Y(y)} = \frac{e^{[-y(\lambda_1 T_1 + \lambda_2 T_2)]}}{E[e^{[-Y(\lambda_1 T_1 + \lambda_2 T_2)]}]} \qquad (12)$$

The Frailty

- The likelihood ratio has the effect of adjusting the probability density of Y as time goes by and as long as no defaults happen there will be more mass for low Y
- The resulting conditional survival probabilities and default intensities are :

$$P[\tau_1 > T_1, \tau_2 > T_2 \mid F_t] = \frac{E[e^{[-T(\lambda_1 T_1 + \lambda_2 T_2)]}]}{E[e^{[-Y(\lambda_1 t + \lambda_2 t)]}]} \qquad (13)$$

$$h_i(t) = \lambda_i \frac{E[e^{[-T(\lambda_1 T_1 + \lambda_2 T_2)]}]}{E[e^{[-Y(\lambda_1 t + \lambda_2 t)]}]} \qquad (14)$$

- Using the Laplace transform of Y and its derivative and plot the hazard rate as a function of time we will see a decreasing behaviour : No defaults are good news for all the rest of the obligors . In a $\Gamma(\alpha,\beta)$ distributed frailty we have

$$h_i(t) = \lambda_i \frac{\alpha\beta}{1 + \beta(\lambda_1 t + \lambda_2 t)} \qquad (15)$$

- Showing a decreasing decay with t similar to an exponential decay .

- **Let us see the case when we have a default of an obligor** . Then we need the conditional density of Y , conditional to observing a default of obligor 1 at time T1
- 2.6 with the density $f_Y(y)$ of Y will gives us the joint density of (τ_1, τ_1,Y) . Then we evaluate the joint density at τ_1=T_1 and divide it by the marginal density to get the conditional density of (τ_1,Y) conditional on $\tau_1 = T_1$. The density of τ_1 can be found by differentiating 2.7 . The result is :

$$f(t_2, y \mid \tau_1 = T_1) = \frac{f(t_1 = T_1, t_2, y)}{\lambda_1 E[Ye^{-Y\lambda_1 T_1}]} = y\lambda_2 e^{-y\lambda_2 t_2} \frac{ye^{-y\lambda_1 T_1}}{E[Ye^{[-Y(\lambda_1 T_1)]}]} f_Y(y) \qquad (16)$$

The Frailty

- For $\tau_1 = T_1$ resulted in a likelihood contribution of $\dfrac{ye^{-y\lambda_1 T_1}}{E[Ye^{[-Y(\lambda_1 T_1)]}]}$ to the joint density .

- If we now integrate out $\tau_2 = T_2$ and divide by its unconditional probability we reach the conditional density of Y conditional on $\tau_1 = T_1$ and $\tau_2 = T_2$:

$$f(y \mid \tau_1 = T_1, \tau_2 > T_2) = \frac{ye^{-y\lambda_1 T_1} e^{-y\lambda_2 T_2}}{E[Ye^{[-Y(\lambda_1 T_1)]} e^{[-Y(\lambda_2 T_2)]}]} f_Y(y) \qquad (17)$$

$$= \underbrace{\frac{e^{-y(\lambda_1 T_1 + \lambda_2 T_2)}}{E[e^{[-Y(\lambda_1 T_1) + Y(\lambda_2 T_2)]}]}}_{\text{No default}} \cdot \underbrace{\frac{y}{E[Y \mid \tau_1 \geq T_1 \wedge \tau_2 \geq T_2]} f_Y(y)}_{\text{Arrival of a default}} \qquad (18)$$

No default — 1

Arrival of a default — 2

- If we now integrate out $\tau_2 = T_2$ and divide by its unconditional probability we reach the conditional density of Y conditional on and

- In equation (18) the likelihood is decomposed as the product of two components : First , the conditioning on survival up to default $\tau_1 >= T_1$ and $\tau_{2>} = T_2$ and the second factor from additionally conditioning on the default time itself $\tau = T_1$. The denominator is just the normalisation of these two contributions .

- If the default had not happened at T_1 we would just have the contribution of the first factor . We have an additional contribution if the default happens and it is proportional to y . Surprise,Surprise .

THE ADDITIONAL DEFAULT CONTRIBUTES TO HIGHER VALUES OF Y IN THE DISTRIBUTION

The Frailty

- Finally if the second obligor defaults inmediately after the first one by (16) and (17) we have :

$$h_2(T_1) = \lambda_2 E[Y \mid \tau_1, \tau_2 > T_1] = \lambda_2 \frac{E[Y^2 e^{-sY}]}{E[Ye^{-sY}]} \qquad (19)$$

- The default intensity just before the default of obligor 1 was :

$$h_2(T_1^-) = \lambda_2 E[Y \mid \tau_1 \geq T_1, \tau_2 > T_1] = \lambda_2 \frac{E[Ye^{-sY}]}{E[Ye^{-sY}]} \qquad (20)$$

- The size of the jump (%) in the default intensity of obligor 2 after a default of obligor 1 at T_1 is :

$$\frac{h_2(T_1) - h_2(T_1^-)}{h_2(T_1^-)} = \frac{Var[Y \mid \tau_1 \geq T_1 \wedge \tau_2 \geq T_1]}{(E[Y \mid \tau_1 \geq T_1 \wedge \tau_2 \geq T_1])^2} \qquad (21)$$

- The size of the relative jump is the square of the coefficient of variation of the frailty (σ_Y / μ_Y)

THE MAGNITUDE OF THE UNCERTAINTY OF Y DETERMINES THE SIZE OF THE INFORMATION BASED CONTAGION

The frailty . Conclusion .

- New defaults of an obligor will have an effect on the frailty towards more risky values .
- The survival of the other obligors becomes worse at the times of default
- This type of contagion is due only to the market information not changes in the creditworthiness of the obligor
- The size of the contagion is the size of the relative jump in the default hazard rate of obligor 2 upon default of obligor 1
- This relative jump is equal to the ratio of variance of Y to the square of the mean of Y . The higher the variance the higher the information that a default has and therefore a higher contagion effect .
- We found that we could separate the contagion jumps from the non-contagion part of the hazard rates .
- Similar jumps can be observed in the copula-based models of default risk . The main strenght and weakness of the copula is the separation of the dependence structure from the marginal survival probabilities . It makes the calibration very easy but different default intensities with high copula dependence will exhibit very unreasonable updating properties(Not good infection properties) .
- As we saw in the properties of the copula based models there is a strong dependance on calendar time and the distance from the initial date .

WE HAVE BETTER MEANS TO HAVE REALISTIC LEVELS OF DEFAULT DEPENDANCE

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Agenda

Evolutionary Portfolio Credit Management

More realistic modelling

- From static models like Vasicek to Gaussian Copula (RiskMetrics)
- Gaussian copula :
  - Higher degrees of freedom
  - May want higher degrees of freedom (t-copula in some cases)
  - Straight forward simulation
  - No analytical tractability
- Updated Dynamic Copula functions or Generalized Archimedean with the introduction of the Frailty model
  - Interesting if Y is $\Gamma(\alpha,\beta)$ distributed (Clayton copula)
  - In general are more tractable
  - More stable
  - Easily interpreted : Y are unobserved stress factors on the default intensities of the individual obligors
  - Contagion effects

More realistic modelling

- The largest difference between the classical approach (Intensity based models) and the Frailty models is the presence of Y that influences the default intensities .
- If all default intensities were perfectly observable then the contagion should not exist and Y should be constant .
- In we could not observe joint influences on defaults of obligors then we should model dependencies with the Copula only .
- The truth is probably somewhere in the middle :
  - Some common influences are observable therefore λ_l should have dependent common factor Y .
  - Some are unobservable therefore modelled using a copula
  - Some are partially observable so contributions to both .

PAYING ATTENTION TO MODEL AND PARAMETER RISKS SHOULD INCREASE THE QUALITY OF INSIGHTS

Conclusions

BEFORE MAKING DECISIONS CHECK MODEL AND PARAMETER RISKS

CHECK UNCERTAINTY AROUND THE RISK RESULTS

CREDIT EXPERIENCE AND INTUITION ARE STILL THE BEST TO MINIMIZE THE ABOVE