Analysis of Hybrid Defaultable Bond Pricing Models

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Analysis of Hybrid Defaultable Bond Pricing Models
Overview

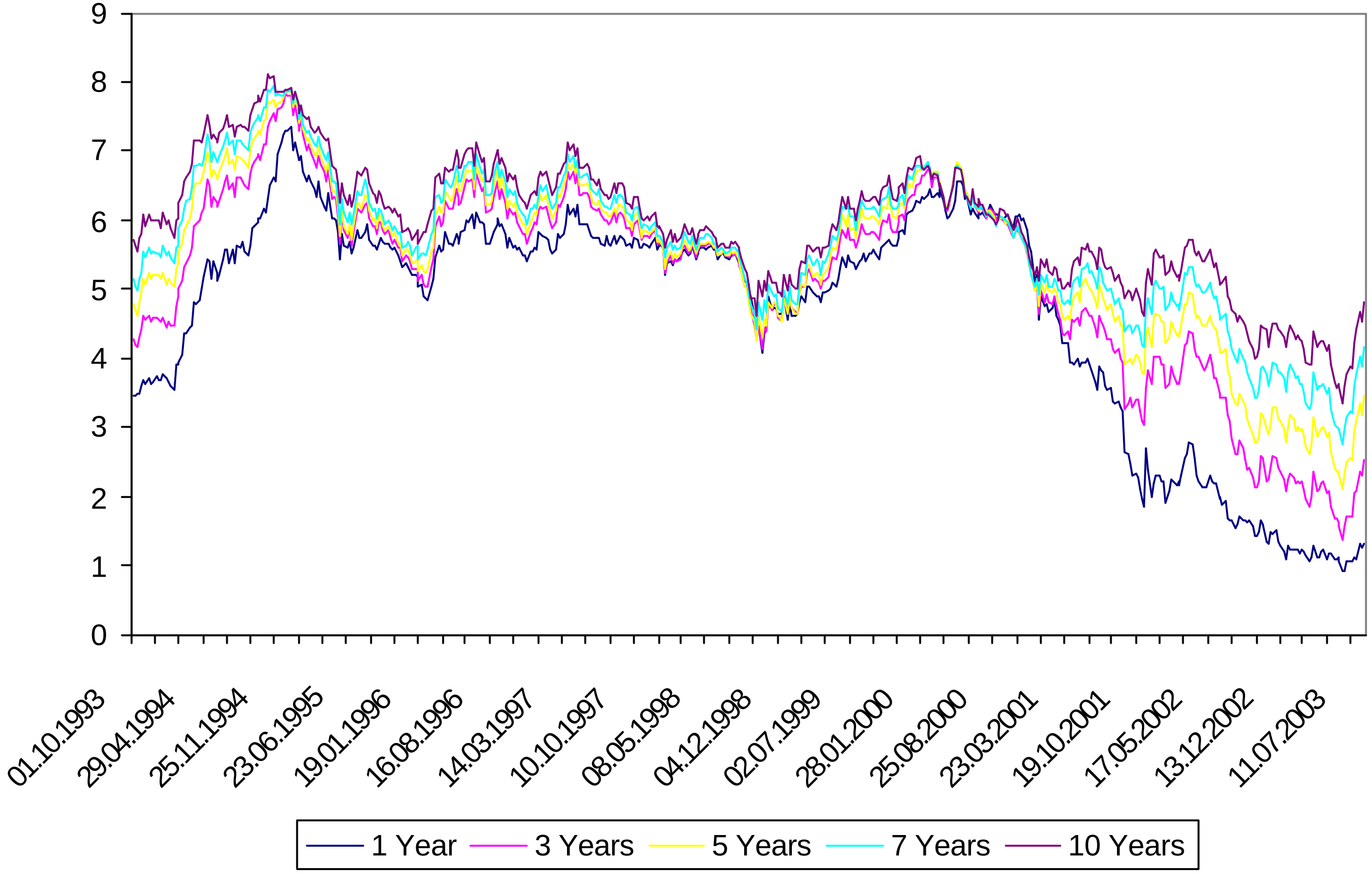
- **The Market Data**
  - Yield Curve Behaviour
    US Treasury Strips
  - Credit Spread Behaviour
    US Industrials A2
  - Credit Spread Behaviour
    US Industrials BBB1
  - Economic Behaviour
    US Gross Domestic Product (GDP)
- **Three Models for the Pricing of Defaultable Bonds**
- **Model Comparison**
- **Further Research**

Treasury and Spread Data
US Market 1993 - 2003

- American Treasury Strips
- A2- and BBB1-rated American Industrials
- Maturities between 3 months and 30 years
- Time series of weekly bond prices from Oktober 8, 1993 to June 1, 2001 (in sample)
- Time series of weekly bond prices from June 8, 2001 to August 15, 2003 (out of sample)
- All prices in US Dollar, i.e. no currency risk in credit spreads
- Parameter estimation using Kalman filters

1- to 10-Year US Treasury Strips in %

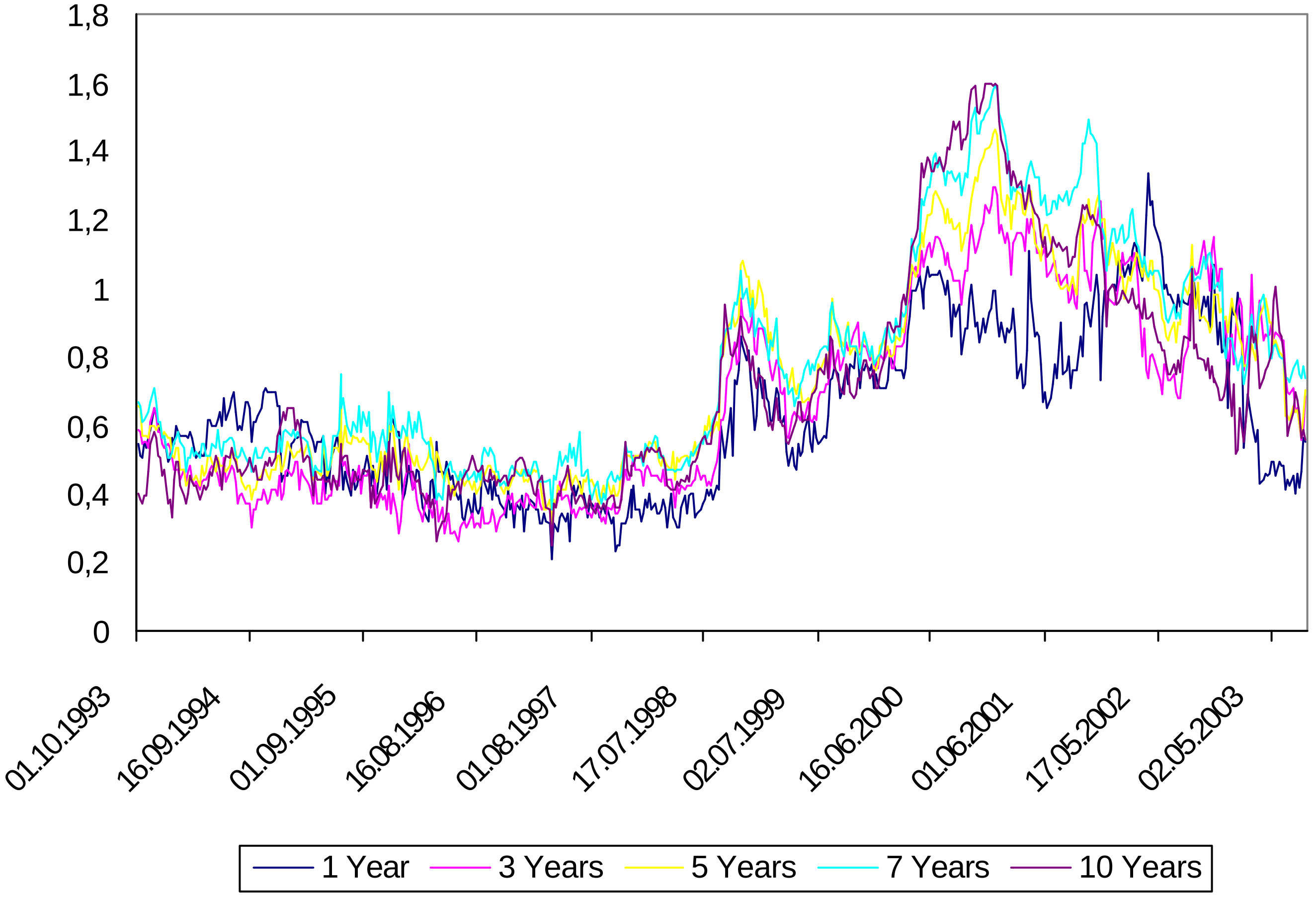
Time Period: 1993 – 2003



Source: Bloomberg

1- To 10 Year Credit Spread of US Industrials A2 in %

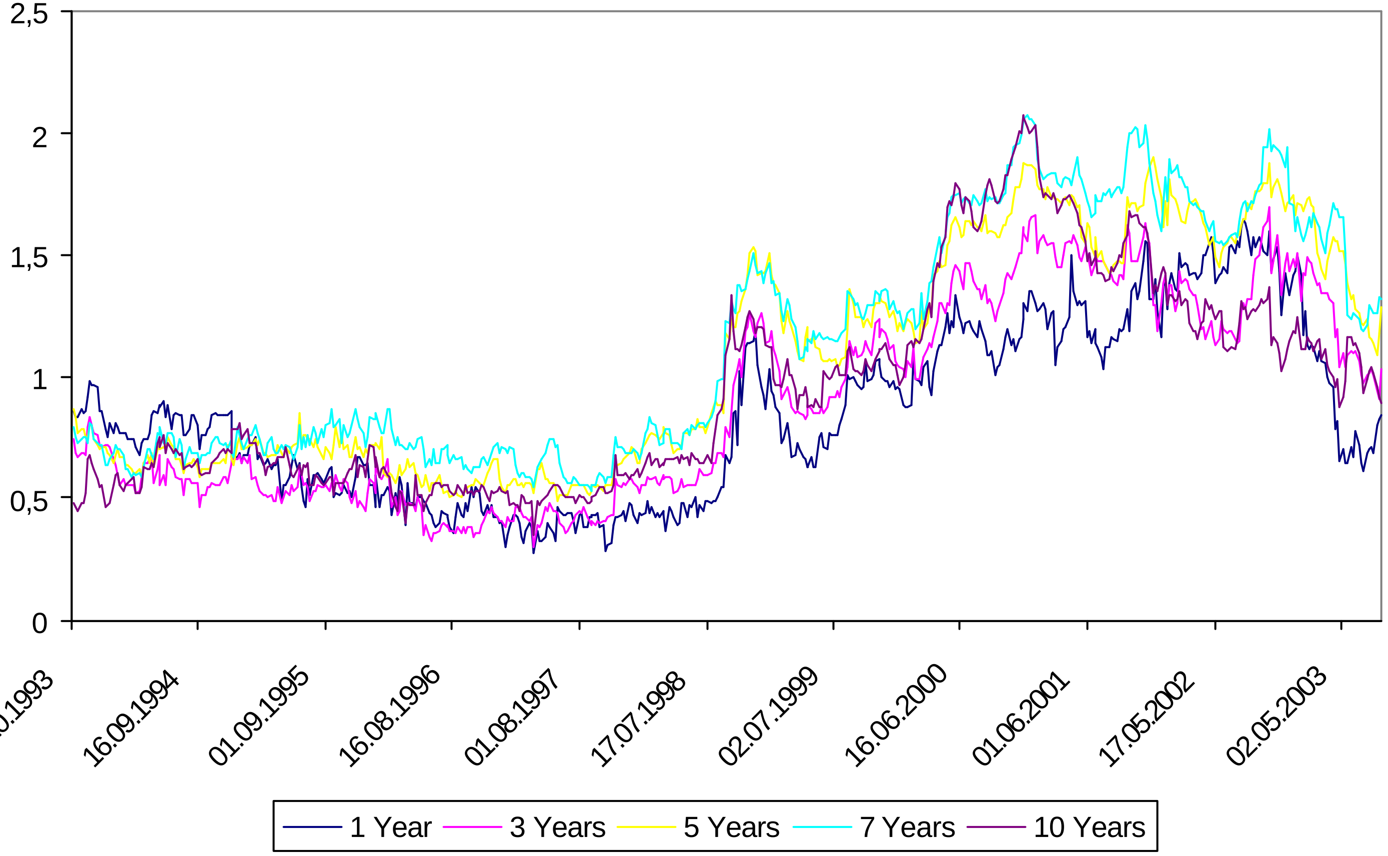
Time Period: 1993 – 2003



Source: Bloomberg

1- To 10 Year Credit Spread of US Industrials BBB1 in %

Time Period: 1993 – 2003



Source: Bloomberg

Generalization of the SZ-Model for the Pricing of Defaultable Bonds
Overview

- **The Market Data**
- **Three Models for the Pricing of Defaultable Bonds**
  - Schmid and Zagst [2000]
  - Bakshi, Madan, Zhang [2001]
  - Generalized Schmid and Zagst [2003]
- **Model Comparison**
- **Further Research**

The Model of Schmid and Zagst [2000]

Modeling of the Stochastic Processes Under the Martingale Measure Q

- Dynamics of the yield curve (non-defaultable short rate)

$$dr(t) = [q_r(t) - a_r \cdot r(t)]dt + s_r dW_r(t), t \in [0, T^*], a_r > 0, s_r > 0$$

- Dynamics of the unobservable counterparty quality (uncertainty index)

$$du(t) = [q_u - a_u \cdot u(t)]dt + s_u \sqrt{u(t)} dW_u(t), t \in [0, T^*], q_u \geq 0, a_u > 0, s_u > 0$$

- Dynamics of the yield spreads (short-rate credit spreads)

$$ds(t) = [b_s \cdot u(t) - a_s \cdot s(t)]dt + s_s \sqrt{s(t)} dW_s(t), t \in [0, T^*], b_s > 0, a_s > 0, s_s > 0$$

- The Wiener processes W_r, W_u, and W_s are uncorrelated

The Model of Schmid and Zagst [2000]
Pricing of Non-Defaultable Bonds

Theorem 1 (Hull and White [1990] , Hull[1997]).
The time t value P(r,t,T) of a non-defaultable zero-coupon bond with maturity T≥t is given by

$$P(r,t,T)=e^{A(t,T)-B(t,T)\cdot r}$$

with

$$B(t,T)=\frac{1}{a_r}\cdot\left(1-e^{-a_r\cdot(T-t)}\right)$$

and

$$A(t,T)=\ln\left(\frac{P(r,0,T)}{P(r,0,t)}\right)-B(t,T)\cdot\frac{\partial\ln P(r,0,t)}{\partial t}-\frac{s_r^2}{4a_r^3}\cdot\left(e^{-a_r\cdot T}-e^{-a_r\cdot t}\right)^2\cdot\left(e^{2\cdot a_r\cdot t}-1\right).$$

The Model of Schmid and Zagst [2000]

Pricing of Defaultable Bonds

Theorem 2 (Schmid and Zagst [2000]).
The time t value $P^d(r,s,u,t,T)$ of a defaultable zero-coupon bond with maturity $T \geq t$ is given by

$$\begin{aligned} P^d(r,s,u,t,T) &= e^{A^d(t,T) - B(t,T)\cdot r - C(t,T)\cdot s - D(t,T)\cdot u} \\ &= P(r,t,T)\cdot e^{A^*(t,T) - C(t,T)\cdot s - D(t,T)\cdot u} \end{aligned}$$

where

$$C(t,T) = \frac{1 - e^{-d_s\cdot(T-t)}}{K_s^1 - K_s^2 \cdot e^{-d_s\cdot(T-t)}} \quad \text{with} \quad d_s = \sqrt{a_s^2 + 2\cdot s_s^2}$$

$$\text{and} \quad K_s^k = \frac{1}{2}\cdot\left(a_s - (-1)^k \cdot d_s\right),\ k \in \{1,2\}$$

$$D(t,T) = -\frac{2\cdot v'(t,T)}{s_u^2 \cdot v(t,T)} \quad \text{with} \quad v(t,T) \text{ complicated}$$

$$A^d(t,T) = A(t,T) + A^*(t,T) \quad \text{with} \quad A^*(t,T) = \frac{2\cdot q_u}{s_u^2}\cdot \ln\left|\frac{v(T,T)}{v(t,T)}\right|$$

The Model of Schmid and Zagst [2000]

Parameter Estimation

Parameter	Estimation Treasury Strips
a_r ($a_{r,real}$)	0.03672 (0.11442)
σ_r (%)	0.83004
real-world mean reversion level for r (%)	6.11

Parameter	Estimation A2	Estimation BBB1
a_s ($a_{s,real}$)	0.94569 (1.19995)	0.96698 (1.34760)
σ_s (%)	0.50424	6.16900
real-world mean reversion level for s (%)	0.41	0.63
θ_u (%)	0.1886	0.03147
a_u ($a_{u,real}$)	0.19613 (0.19990)	0.00710 (0.04683)
σ_u (%)	0.11206	3.63922
real-world mean reversion level for u (%)	0.94	0.67

The Model of Schmid and Zagst [2000]

Linear Regression of Model vs. Empirical Treasury Strips*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-1.65553*10^{-5}$ ($1.86057*10^{-6}$)	0.81293 (0.81693)	0.6985 (0.7500)
3 Years	$-9.28756*10^{-6}$ ($-4.28053*10^{-6}$)	1.02622 (1.03008)	0.9442 (0.9560)
5 Years	$-1.34277*10^{-6}$ ($-2.68422*10^{-6}$)	1.09104 (1.08467)	0.9872 (0.9926)
7 Years	$4.73147*10^{-6}$ ($1.03906*10^{-6}$)	1.09526 (1.08400)	0.9513 (0.9606)
10 Years	$6.51812*10^{-6}$ ($-1.95064*10^{-6}$)	1.08763 (1.07958)	0.9211 (0.9404)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$-1.25500*10^{-4}$ ($-7.80711*10^{-5}$)	0.71893 (0.79067)	0.6162 (0.7060)
3 Years	$-4.89500*10^{-5}$ ($-2.99150*10^{-5}$)	1.03726 (0.99671)	0.9139 (0.9294)
5 Years	$1.03988*10^{-5}$ ($1.96055*10^{-5}$)	1.09909 (1.08601)	0.9917 (0.9921)
7 Years	$3.90855*10^{-5}$ ($4.39230*10^{-5}$)	1.06966 (1.09057)	0.9633 (0.9729)
10 Years	$6.27278*10^{-5}$ ($4.73641*10^{-5}$)	1.03269 (1.07702)	0.9180 (0.9482)

* Values in brackets are without 2.5% of max. outliers

The Model of Schmid and Zagst [2000]

Linear Regression of A2 - Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-4.40045*10^{-6}$ $(-1.00039*10^{-5})$	1.49338 (1.38773)	0.6571 (0.6690)
3 Years	$-4.34480*10^{-6}$ $(5.21923*10^{-6})$	1.43305 (1.36455)	0.4523 (0.4622)
5 Years	$-8.60220*10^{-6}$ $(-2.17228*10^{-6})$	1.68180 (1.53227)	0.3904 (0.3816)
7 Years	$-6.22185*10^{-6}$ $(-3.26028*10^{-6})$	1.77772 (1.55750)	0.2930 (0.2745)
10 Years	$2.34010*10^{-6}$ $(-1.89566*10^{-6})$	1.64530 (1.51296)	0.1682 (0.1680)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$2.64323*10^{-6}$ $(-9.38798*10^{-6})$	1.48722 (1.35270)	0.6571 (0.6411)
3 Years	$2.72865*10^{-5}$ $(5.33658*10^{-6})$	1.72891 (1.46988)	0.3975 (0.3304)
5 Years	$1.34177*10^{-5}$ $(1.86122*10^{-5})$	1.49348 (1.28500)	0.4404 (0.4103)
7 Years	$-1.27622*10^{-5}$ $(2.28318*10^{-5})$	0.80991 (0.69241)	0.1215 (0.1289)
10 Years	$-2.28218*10^{-5}$ $(-5.22126*10^{-6})$	0.80043 (0.65427)	0.0437 (0.0406)

* Values in brackets are without 2.5% of max. outliers

The Model of Schmid and Zagst [2000]

Linear Regression of BBB1 - Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-5.14297 \cdot 10^{-6}$ ($-7.51586 \cdot 10^{-6}$)	1.50472 (1.41491)	0.8622 (0.8503)
3 Years	$-3.06176 \cdot 10^{-6}$ ($6.50870 \cdot 10^{-6}$)	1.14942 (1.11749)	0.5321 (0.5643)
5 Years	$-9.55252 \cdot 10^{-7}$ ($3.34430 \cdot 10^{-6}$)	0.87634 (0.77505)	0.2893 (0.2968)
7 Years	$-1.34109 \cdot 10^{-7}$ ($-9.50104 \cdot 10^{-6}$)	0.94840 (0.92787)	0.3315 (0.3489)
10 Years	$-7.31040 \cdot 10^{-7}$ ($1.008086 \cdot 10^{-5}$)	1.04541 (1.09752)	0.3538 (0.4397)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$7.43331 \cdot 10^{-6}$ ($8.94970 \cdot 10^{-6}$)	1.43015 (1.37382)	0.9290 (0.9273)
3 Years	$1.19113 \cdot 10^{-5}$ ($2.24141 \cdot 10^{-5}$)	1.40258 (1.46189)	0.5975 (0.7150)
5 Years	$2.83098 \cdot 10^{-6}$ ($1.11236 \cdot 10^{-5}$)	0.69734 (0.58717)	0.2088 (0.1912)
7 Years	$5.77097 \cdot 10^{-6}$ ($3.19880 \cdot 10^{-5}$)	0.95732 (0.88070)	0.3444 (0.4228)
10 Years	$-1.57533 \cdot 10^{-5}$ ($-2.93676 \cdot 10^{-5}$)	0.91349 (0.93195)	0.3612 (0.4532)

* Values in brackets are without 2.5% of max. outliers

The Model of Schmid and Zagst [2000]
Correlations

A2	dW_s	dW_u	dW_r
dW_s	1	-0.05413	-0.13693
dW_u	-0.05413	1	-0.22585
dW_r	-0.13693	-0.22585	1

BBB1	dW_s	dW_u	dW_r
dW_s	1	0.03749	-0.19295
dW_u	0.03749	1	-0.18097
dW_r	-0.19295	-0.18097	1

Generalization of the SZ-Model for the Pricing of Defaultable Bonds

Overview

- **The Market Data**
- **Three Models for the Pricing of Defaultable Bonds**
  - Schmid and Zagst [2000]
  - Bakshi, Madan, Zhang [2001]
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- **Model Comparison**
- **Further Research**

The Model of Bakshi, Madan, Zhang [2001]

Modeling of the Stochastic Processes Under the Martingale Measure Q

- **Dynamics of the yield curve (non-defaultable short rate)**

$$dr(t) = [w(t) - a_r \cdot r(t)]\,dt + s_r dW_r(t),\ t \in [0, T^*],\ a_r > 0,\ s_r > 0$$

- **Dynamics of the unobservable drift driving process**

$$dw(t) = [q_w - a_w \cdot w(t)]\,dt + s_w \cdot \left(r_{rw} dW_r(t) + \sqrt{1 - r_{rw}^2}\, dW_w(t)\right),\ t \in [0, T^*],\ q_w \geq 0,\ a_w > 0,\ s_w > 0$$

- **Dynamics of the unobservable uncertainty index**

$$du(t) = [q_u - a_u \cdot u(t)]\,dt + s_u \cdot \left(r_{ru} dW_r(t) + \sqrt{1 - r_{ru}^2}\, dW_u(t)\right),\ t \in [0, T^*],\ q_u \geq 0,\ a_u > 0,\ s_u > 0$$

- **The Wiener processes W_r, $W_?$, and W_u are uncorrelated**
- **The short rate spread is given by**

$$s(t) = \Lambda_0 + (\Lambda_r - 1) \cdot r(t) + \Lambda_u \cdot u(t),\ t \in [0, T^*],\ \Lambda_0, \Lambda_r, \Lambda_u \in IR, \text{ i.e.}$$

$$ds(t) = [q_s + b_{sw} \cdot w(t) + b_{su} \cdot u(t) - a_s \cdot s(t)]\,dt + s_s\, dW_s(t)$$

with $q_s := \Lambda_0 \cdot a_r + \Lambda_u \cdot q_u$, $b_{su} := \Lambda_u \cdot (a_r - a_u)$, $b_{sw} := \Lambda_r - 1$, $a_s = a_r$,

and $s_s\, dW_s(t) := ((\Lambda_r - 1) \cdot s_r + \Lambda_u \cdot s_u \cdot r_{ru})\, dW_r(t) + \Lambda_u \cdot s_u \cdot \sqrt{1 - r_{ru}^2}\, dW_u(t)$

The Model of Bakshi, Madan, Zhang [2001]

Pricing of Non-Defaultable Bonds

Theorem 5.

The time t value P(r,? ,t,T) of a non-defaultable zero-coupon bond with maturity T≥t is given by

$$P(r,w,t,T) = e^{A(t,T) - B(t,T)\cdot r - E(t,T)\cdot w}$$

with

$$B(t,T) = \frac{1}{a_r}\cdot\left(1 - e^{-a_r\cdot(T-t)}\right),\quad E(t,T) = \frac{1}{a_r}\cdot\left(\frac{1-e^{-a_w\cdot(T-t)}}{a_w} + \frac{e^{-a_w\cdot(T-t)} - e^{-a_r\cdot(T-t)}}{a_w - a_r}\right),$$

and

$$A(t,T) = \int_t^T \frac{s_r^2}{2}\cdot B^2(t,T) + \frac{s_w^2}{2}\cdot E^2(t,T) + r_{rw}\cdot s_r\cdot s_w\cdot B(t,T)\cdot E(t,T) - q_w\cdot E(t,T)\,dt.$$

The Model of Bakshi, Madan, Zhang [2001]
Pricing of Non-Defaultable Bonds

Theorem 6.
The time t value P^d(r,u,? ,t,T) of a defaultable zero-coupon bond with maturity T≥t is given by

$$P^d(r,u,w,t,T) = e^{A^d(t,T) - B^d(t,T)\cdot r - D(t,T)\cdot u - E^d(t,T)\cdot w} = P(r,w,t,T)\cdot e^{A^*(t,T) - (\Lambda_r - 1)\cdot B(t,T)\cdot r - D(t,T)\cdot u - (\Lambda_r - 1)\cdot E(t,T)\cdot w}$$

where

$$B^d(t,T) = \Lambda_r \cdot B(t,T),\ \ D(t,T) = \frac{\Lambda_u}{a_u}\cdot\left(1 - e^{-a_u\cdot(T-t)}\right),$$

$$E^d(t,T) = \Lambda_r \cdot E(t,T),\ \ A^*(t,T) = A^d(t,T) - A(t,T),$$

$$\begin{aligned} A^d(t,T) \;=\; & \int_t^T \frac{s_r^2}{2}\cdot\Lambda_r^2\cdot B^2(t,T) + \frac{s_u^2}{2}\cdot D^2(t,T) + \frac{s_w^2}{2}\cdot\Lambda_r^2\cdot E^2(t,T)\,dt \\ & + \int_t^T s_r\cdot s_u\cdot r_{ru}\cdot\Lambda_r\cdot B(t,T)\cdot D(t,T) + s_r\cdot s_w\cdot r_{rw}\cdot\Lambda_r^2\cdot B(t,T)\cdot E(t,T)\,dt \\ & - \int_t^T q_u\cdot D(t,T) + q_w\cdot\Lambda_r\cdot E(t,T) - \Lambda_0\,dt \end{aligned}$$

The Model of Bakshi, Madan, Zhang [2001]

Parameter Estimation

Parameter	Estimation Treasury
a_r ($a_{r,real}$)	0.03939 (0.10693)
r_{rw}	-0.31623
σ_r (%)	0.11198
real-world mean reversion for r (%)	6.14
θ_w(%)	0.12422
a_w ($a_{w,real}$)	0.30486 (0.18920)
σ_w(%)	0.37280
real-world mean reversion for w (%)	0.65

Parameter	Estimation A2	Estimation BBB1
θ_u (%)	0.05256	0.06119
a_u ($a_{u,real}$)	$1.47407*10^{-6}$ (0.64952)	$1.73866*10^{-6}$ (0.76055)
r_{ru}	0.63164	0.60431
σ_u (%)	0.22824	0.26460
real-world mean reversion for u (%)	0.08092	0.08046
L_0	0.01390	0.01640
L_r	0.85413	0.84470
real-world mean reversion for short spread (%)	0.58	0.77

The Model of Bakshi, Madan, Zhang [2001]

Linear Regression of Model vs. Empirical Treasury Strips*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-3.42806 \cdot 10^{-7}$ ($6.44208 \cdot 10^{-6}$)	0.85830 (0.88428)	0.82061 (0.85126)
3 Years	$1.82370 \cdot 10^{-6}$ ($-6.71713 \cdot 10^{-7}$)	1.01426 (1.01951)	0.97020 (0.97785)
5 Years	$1.63023 \cdot 10^{-6}$ ($2.41639 \cdot 10^{-6}$)	1.02874 (1.02707)	0.97977 (0.98379)
7 Years	$6.02062 \cdot 10^{-7}$ ($-1.14269 \cdot 10^{-6}$)	1.02421 (1.01974)	0.98871 (0.99105)
10 Years	$-4.97210 \cdot 10^{-6}$ ($-5.23195 \cdot 10^{-6}$)	0.98623 (0.97926)	0.97120 (0.97855)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$-3.70868 \cdot 10^{-5}$ ($-4.47953 \cdot 10^{-5}$)	0.72644 (0.79991)	0.66948 (0.75295)
3 Years	$1.98333 \cdot 10^{-6}$ ($1.49782 \cdot 10^{-5}$)	1.02725 (1.02631)	0.88702 (0.91691)
5 Years	$5.76821 \cdot 10^{-6}$ ($1.12280 \cdot 10^{-5}$)	1.06261 (1.05107)	0.97391 (0.97736)
7 Years	$7.52360 \cdot 10^{-6}$ ($1.17185 \cdot 10^{-5}$)	1.01586 (1.01912)	0.98858 (0.99034)
10 Years	$1.48360 \cdot 10^{-6}$ ($-6.25209 \cdot 10^{-6}$)	0.94568 (0.95998)	0.97137 (0.97781)

* Values in brackets are without 2.5% of max. outliers

The Model of Bakshi, Madan, Zhang [2001]

Linear Regression of A2-Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-6.66014 \cdot 10^{-6}$ ($-7.81042*10^{-6}$)	0.95710 (0.95283)	0.22010 (0.27764)
3 Years	$2.30325 \cdot 10^{-7}$ ($3.63113*10^{-6}$)	1.05105 (1.04419)	0.48248 (0.53328)
5 Years	$1.87651 \cdot 10^{-6}$ ($6.32987*10^{-6}$)	0.87247 (0.82517)	0.33789 (0.36007)
7 Years	$4.28036 \cdot 10^{-6}$ ($4.69386*10^{-6}$)	0.82443 (0.85134)	0.28920 (0.34767)
10 Years	$9.89073 \cdot 10^{-6}$ ($1.78723*10^{-5}$)	0.69080 (0.74295)	0.21149 (0.27725)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$3.34013*10^{?6}$ ($2.139434*10^{-5}$)	0.72319 (0.59840)	0.11174 (0.10146)
3 Years	$1.25743*10^{?5}$ ($3.048757*10^{-5}$)	1.43400 (1.45537)	0.48341 (0.53835)
5 Years	$-3.60543*10^{?6}$ ($2.699827*10^{-5}$)	0.90181 (0.85606)	0.39617 (0.44035)
7 Years	$-1.82706*10^{?5}$ ($-2.873219 \cdot 10^{-6}$)	0.59642 (0.53496)	0.20386 (0.20944)
10 Years	$-1.66908*10^{?5}$ ($4.191887*10^{-5}$)	0.68768 (0.75671)	0.14285 (0.22991)

* Values in brackets are without 2.5% of max. outliers

The Model of Bakshi, Madan, Zhang [2001]

Linear Regression of BBB1-Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-9.66122*10^{-6}$ ($-1.03797*10^{-5}$)	0.99705 (0.96803)	0.24719 (0.29728)
3 Years	$-1.35793*10^{-6}$ ($4.22798*10^{-9}$)	1.07163 (1.08342)	0.53206 (0.59715)
5 Years	$6.53142*10^{-6}$ ($7.26902*10^{-6}$)	0.65241 (0.61264)	0.20462 (0.22179)
7 Years	$1.06134*10^{-5}$ ($-3.83532*10^{-6}$)	0.63931 (0.70004)	0.19620 (0.24519)
10 Years	$1.19527*10^{-5}$ ($1.79937*10^{-5}$)	0.69952 (0.73303)	0.20916 (0.27495)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$-6.51196*10^{-6}$ ($1.40783*10^{-5}$)	0.95707 (0.96168)	0.23218 (0.28610)
3 Years	$1.688527*10^{-5}$ ($2.48936*10^{-5}$)	1.46138 (1.35229)	0.56245 (0.58531)
5 Years	$5.21240*10^{-6}$ ($1.38297*10^{-5}$)	0.74805 (0.64743)	0.22002 (0.21269)
7 Years	$5.23919*10^{-6}$ ($3.34496*10^{-5}$)	0.74985 (0.72745)	0.19756 (0.27076)
10 Years	$-8.81659*10^{-6}$ ($-2.24224*10^{-5}$)	0.81042 (0.80810)	0.26765 (0.31975)

* Values in brackets are without 2.5% of max. outliers

The Model of Bakshi, Madan, Zhang [2001]

Correlations

A2	dW_u	dW_w	dW_r
dW_u	1	0.17103	0.32827
dW_w	0.17103	1	0.15979
dW_r	0.32827	0.15979	1

BBB1	dW_u	dW_w	dW_r
dW_u	1	0.16431	0.30451
dW_w	0.16431	1	0.15979
dW_r	0.30451	0.15979	1

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- **Model Comparison**
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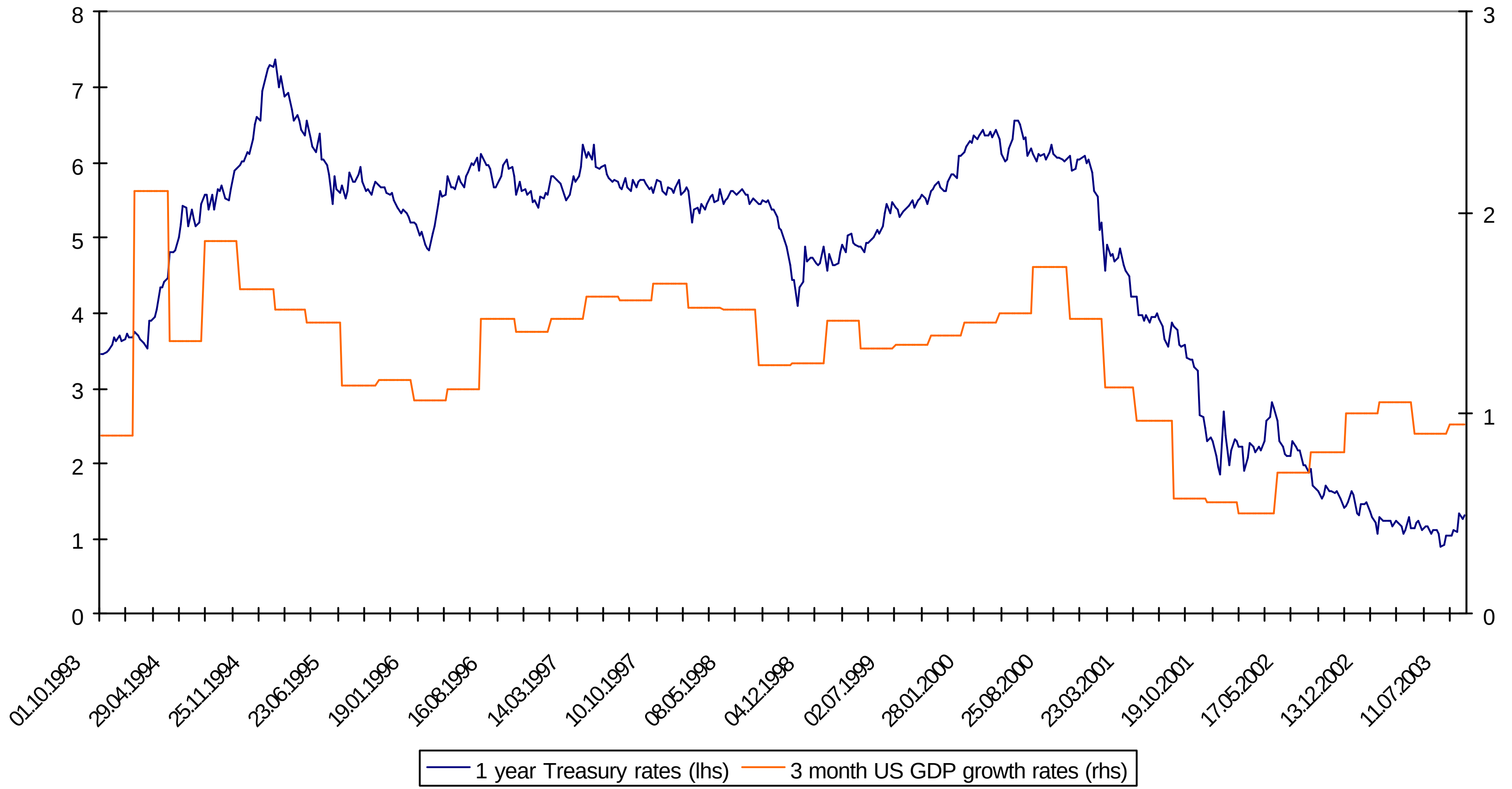
US Treasury Strips and Industrials vs. US Gross Domestic Product

Time Period: 1993 – 2003

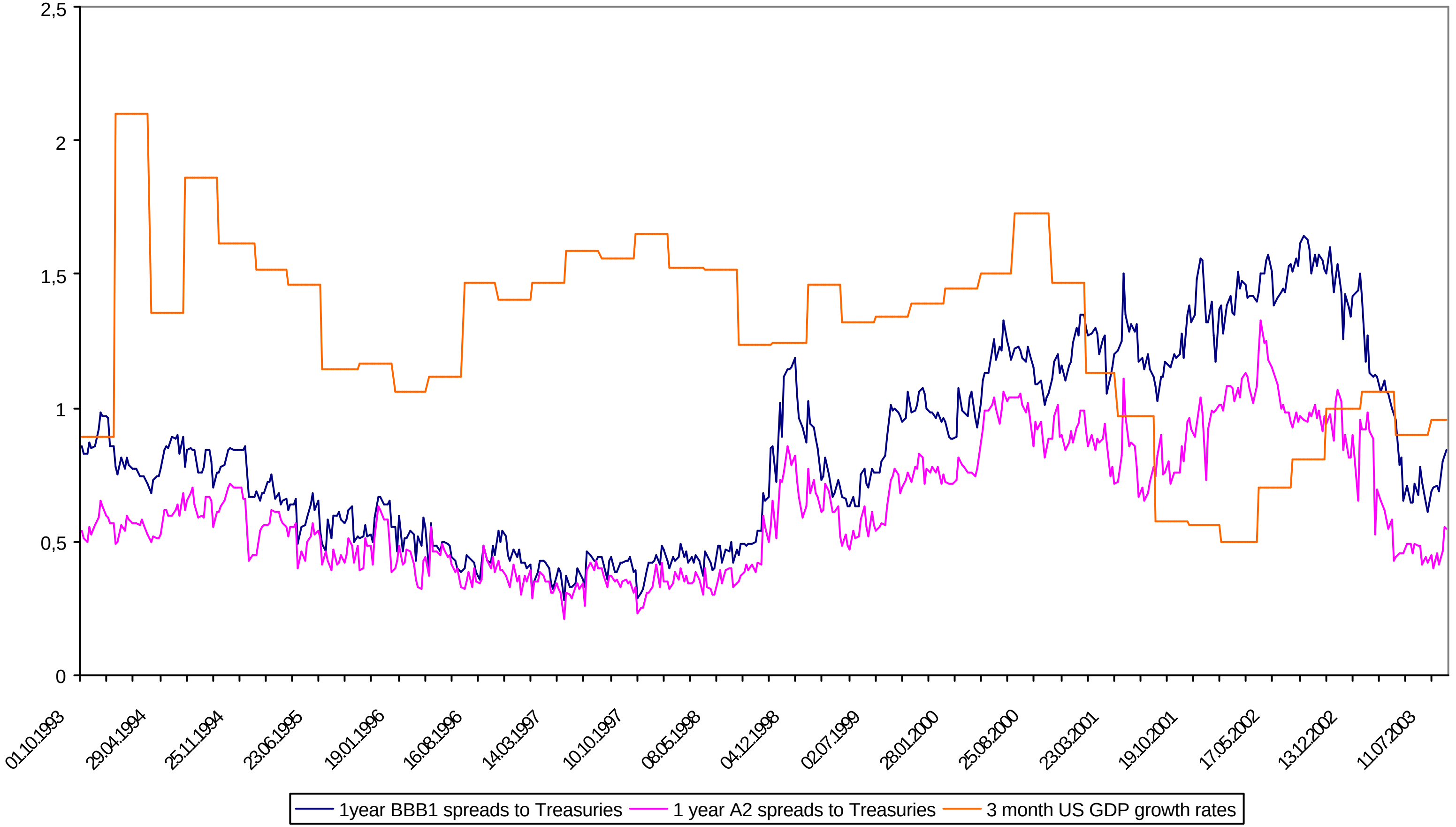


Source: Bloomberg

Linear Regression: $R(t+0.25,t+1.25) = -0.62 + 0.84 \cdot R(t,t+1) + 1.03 \cdot w(t) + \varepsilon$

US Credit Spreads vs. US Gross Domestic Product

Time Period: 1993 – 2003



Source: Bloomberg

Linear Regression:

$$S^{A2}(t+0.25,t+1.25) = 0.25 + 0.84 \cdot S^{A2}(t,t+1) - 0.10 \cdot w(t) + \varepsilon$$

$$S^{BBB1}(t+0.25,t+1.25) = 0.24 + 0.77 \cdot S^{BBB1}(t,t+1) - 0.08 \cdot w(t) + \varepsilon$$

The Generalized Model of Schmid and Zagst [2003]

Modeling of the Stochastic Processes Under the Martingale Measure Q

- **Dynamics of the yield curve (non-defaultable short rate)**

$$dr(t) = [q_r(t) + b_{rw} \cdot w(t) - a_r \cdot r(t)]dt + s_r dW_r(t), t \in [0, T^*], a_r > 0, s_r > 0$$

- **Dynamics of the economy index**

$$dw(t) = [q_w - a_w \cdot w(t)]dt + s_w dW_w(t), t \in [0, T^*], q_w \geq 0, a_w > 0, s_w > 0$$

- **Dynamics of the uncertainty index**

$$du(t) = [q_u - a_u \cdot u(t)]dt + s_u dW_u(t), t \in [0, T^*], q_u \geq 0, a_u > 0, s_u > 0$$

- **Dynamics of the yield spread (short-rate credit spread)**

$$ds(t) = [q_s + b_{su} \cdot u(t) - b_{sw} \cdot w(t) - a_s \cdot s(t)]dt + s_s dW_s(t), t \in [0, T^*], \quad ?_s \geq 0, b_{su} > 0, b_{sw} > 0, a_s > 0, s_s > 0$$

- **The Wiener processes W_r , $W_?$, W_u , and W_s are uncorrelated**

The Generalized Model of Schmid and Zagst [2003]

Pricing of Non-Defaultable Bonds

Theorem 7.

The time t value P(r,? ,t,T) of a non-defaultable zero-coupon bond with maturity T≥t is given by

$$P(r,w,t,T) = e^{A(t,T) - B(t,T)\cdot r - E(t,T)\cdot w}$$

with

$$B(t,T) = \frac{1}{a_r}\cdot\left(1 - e^{-a_r\cdot(T-t)}\right),\quad E(t,T) = \frac{b_{rw}}{a_r}\cdot\left(\frac{1-e^{-a_w\cdot(T-t)}}{a_w} + \frac{e^{-a_w\cdot(T-t)} - e^{-a_r\cdot(T-t)}}{a_w - a_r}\right),$$

and

$$A(t,T) = \int_t^T \frac{s_r^2}{2}\cdot B^2(t,T) + \frac{s_w^2}{2}\cdot E^2(t,T) - q_r(t)\cdot B(t,T) - q_w\cdot E(t,T)\,dt\,.$$

The Generalized Model of Schmid and Zagst [2003]

Pricing of Non-Defaultable Bonds

Theorem 8.

The time t value P^d(r,s,u,? ,t,T) of a defaultable zero-coupon bond with maturity T≥t is given by

$$\begin{aligned} P^d(r,s,u,w,t,T) &= e^{A^d(t,T)-B(t,T)\cdot r-C(t,T)\cdot s-D(t,T)\cdot u-E^d(t,T)\cdot w} \\ &= P(r,w,t,T)\cdot e^{A^*(t,T)-C(t,T)\cdot s-D(t,T)\cdot u+E^*(t,T)\cdot w} \end{aligned}$$

where

$$C(t,T)=\frac{1}{a_s}\cdot\left(1-e^{-a_s\cdot(T-t)}\right),\ D(t,T)=\frac{b_{su}}{a_s}\cdot\left(\frac{1-e^{-a_u\cdot(T-t)}}{a_u}+\frac{e^{-a_u\cdot(T-t)}-e^{-a_s\cdot(T-t)}}{a_u-a_s}\right),$$

$$E^d(t,T)=E(t,T)-E^*(t,T)\ \text{with}\ E^*(t,T)=\frac{b_{sw}}{a_s}\cdot\left(\frac{1-e^{-a_w\cdot(T-t)}}{a_w}+\frac{e^{-a_w\cdot(T-t)}-e^{-a_s\cdot(T-t)}}{a_w-a_s}\right),$$

$$A^d(t,T)=A(t,T)+A^*(t,T)\ \text{with}$$

$$\begin{aligned} A^*(t,T) &= \int_t^T \frac{s_s^2}{2}\cdot C^2(t,T)+\frac{s_u^2}{2}\cdot D^2(t,T)+\frac{s_w^2}{2}\cdot E^{*2}(t,T)-s_w^2\cdot E(t,T)\cdot E^*(t,T)\,dt \\ &\quad -\int_t^T q_s\cdot C(t,T)+q_u\cdot D(t,T)-q_w\cdot E^*(t,T)\,dt \end{aligned}$$

The Generalized Model of Schmid and Zagst [2003]

Linear Regression of Model vs. Empirical Treasury Strips*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-1.65594*10^{-5}$ ($1.84746*10^{-6}$)	0.81332 (0.81736)	0.69841 (0.74989)
3 Years	$-9.28521*10^{-6}$ ($-4.28254*10^{-6}$)	1.02675 (1.03062)	0.58185 (0.95598)
5 Years	$-1.33401*10^{-6}$ ($-2.68575*10^{-6}$)	1.09157 (1.08519)	0.98710 (0.99252)
7 Years	$4.74445*10^{-6}$ ($1.07556*10^{-6}$)	1.09586 (1.08462)	0.95131 (0.96057)
10 Years	$6.53557*10^{-6}$ ($-1.91628*10^{-6}$)	1.08823 (1.08019)	0.92111 (0.94044)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$-1.30993*10^{-4}$ ($-8.83878*10^{-5}$)	0.71971 (0.79157)	0.61501 (0.70458)
3 Years	$-4.47168*10^{-5}$ ($-2.58655*10^{-5}$)	1.03796 (0.99727)	0.91318 (0.92864)
5 Years	$9.92528*10^{-6}$ ($1.90114*10^{-5}$)	1.10020 (1.08714)	0.99165 (0.99200)
7 Years	$4.42484*10^{-5}$ ($4.92335*10^{-5}$)	1.07035 (1.09116)	0.96226 (0.97177)
10 Years	$6.57625*10^{-5}$ ($5.07120*10^{-5}$)	1.03363 (1.07789)	0.91782 (0.94797)

* Values in brackets are without 2.5% of max. outliers

The Generalized Model of Schmid and Zagst [2003]

Linear Regression of A2-Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-4.78155 \cdot 10^{-6}$ ($-9.25154 \cdot 10^{-6}$)	1.67240 (1.55101)	0.69999 (0.69755)
3 Years	$-1.90116 \cdot 10^{-6}$ ($5.48928 \cdot 10^{-8}$)	1.28948 (1.22758)	0.49129 (0.49990)
5 Years	$-6.72453 \cdot 10^{-6}$ ($-4.71346 \cdot 10^{-6}$)	1.39512 (1.35589)	0.56484 (0.57421)
7 Years	$-6.28633 \cdot 10^{-6}$ ($1.26624 \cdot 10^{-7}$)	1.39467 (1.35121)	0.55042 (0.57212)
10 Years	$1.84827 \cdot 10^{-6}$ ($1.12954 \cdot 10^{-5}$)	1.06842 (1.10605)	0.34770 (0.41272)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$1.96768 \cdot 10^{-5}$ ($1.94970 \cdot 10^{-5}$)	1.81906 (1.63470)	0.72218 (0.62540)
3 Years	$2.21200 \cdot 10^{-5}$ ($1.05427 \cdot 10^{-6}$)	1.74768 (1.48103)	0.41918 (0.33666)
5 Years	$1.83691 \cdot 10^{-5}$ ($2.19037 \cdot 10^{-5}$)	1.48593 (1.30226)	0.60484 (0.56088)
7 Years	$1.03026 \cdot 10^{-6}$ ($1.13412 \cdot 10^{-5}$)	1.09252 (1.01194)	0.40056 (0.43584)
10 Years	$-1.15402 \cdot 10^{-5}$ ($4.21980 \cdot 10^{-5}$)	0.89053 (0.91302)	0.14868 (0.20619)

* Values in brackets are without 2.5% of max. outliers

The Generalized Model of Schmid and Zagst [2003]

Linear Regression of BBB1-Model vs. Empirical Credit Spreads*

In sample:

Time to Maturity	a	b	R^2
1 Year	$-7.03927 \cdot 10^{-6}$ ($-4.65670 \cdot 10^{-6}$)	1.66696 (1.54556)	0.79520 (0.78569)
3 Years	$-5.01540 \cdot 10^{-6}$ ($3.28504 \cdot 10^{-6}$)	1.26562 (1.22384)	0.47886 (0.50129)
5 Years	$-1.22915 \cdot 10^{-6}$ ($-1.57571 \cdot 10^{-6}$)	0.90122 (0.75502)	0.22806 (0.21666)
7 Years	$-1.39494 \cdot 10^{-6}$ ($-6.36937 \cdot 10^{-6}$)	1.00711 (0.99147)	0.28241 (0.30946)
10 Years	$-4.57164 \cdot 10^{-6}$ ($6.53896 \cdot 10^{-6}$)	1.18392 (1.31260)	0.35411 (0.46737)

Out of sample:

Time to Maturity	a	b	R^2
1 Year	$2.56815 \cdot 10^{-5}$ ($2.62905 \cdot 10^{-5}$)	1.73378 (1.64262)	0.84001 (0.84160)
3 Years	$2.80024 \cdot 10^{-5}$ ($3.21236 \cdot 10^{-5}$)	1.67596 (1.60898)	0.50437 (0.53628)
5 Years	$6.26282 \cdot 10^{-6}$ ($-2.09998 \cdot 10^{-5}$)	0.82490 (0.67883)	0.16261 (0.14651)
7 Years	$1.98231 \cdot 10^{-5}$ ($4.54770 \cdot 10^{-5}$)	1.20134 (1.15741)	0.29936 (0.40931)
10 Years	$6.70939 \cdot 10^{-7}$ ($5.23056 \cdot 10^{-6}$)	1.21837 (1.07502)	0.35553 (0.37012)

* Values in brackets are without 2.5% of max. outliers

The Generalized Model of Schmid and Zagst [2003]

Correlations

A2	dW_s	dW_u	dW_r	dW_ω
dW_s	1	0.14028	-0.12933	0.55907
dW_u	0.14028	1	-0.18696	-0.01855
dW_r	-0.12933	-0.18696	1	0.30271
dW_ω	0.55907	-0.01855	0.30271	1

BBB1	dW_s	dW_u	dW_r	dW_ω
dW_s	1	0.11551	-0.16967	0.38340
dW_u	0.11551	1	-0.15621	0.14069
dW_r	-0.16967	-0.15621	1	0.30271
dW_ω	0.38340	0.14069	0.30271	1

Generalization of the SZ-Model for the Pricing of Defaultable Bonds

Overview

- **Lessons from the Market**
- **Three Models for the Pricing of Defaultable Bonds**
- **Model Comparison**
- **Further Research**

Model Comparison

Average Absolute Deviation of Treasury Strips, Credit Spreads and R^2

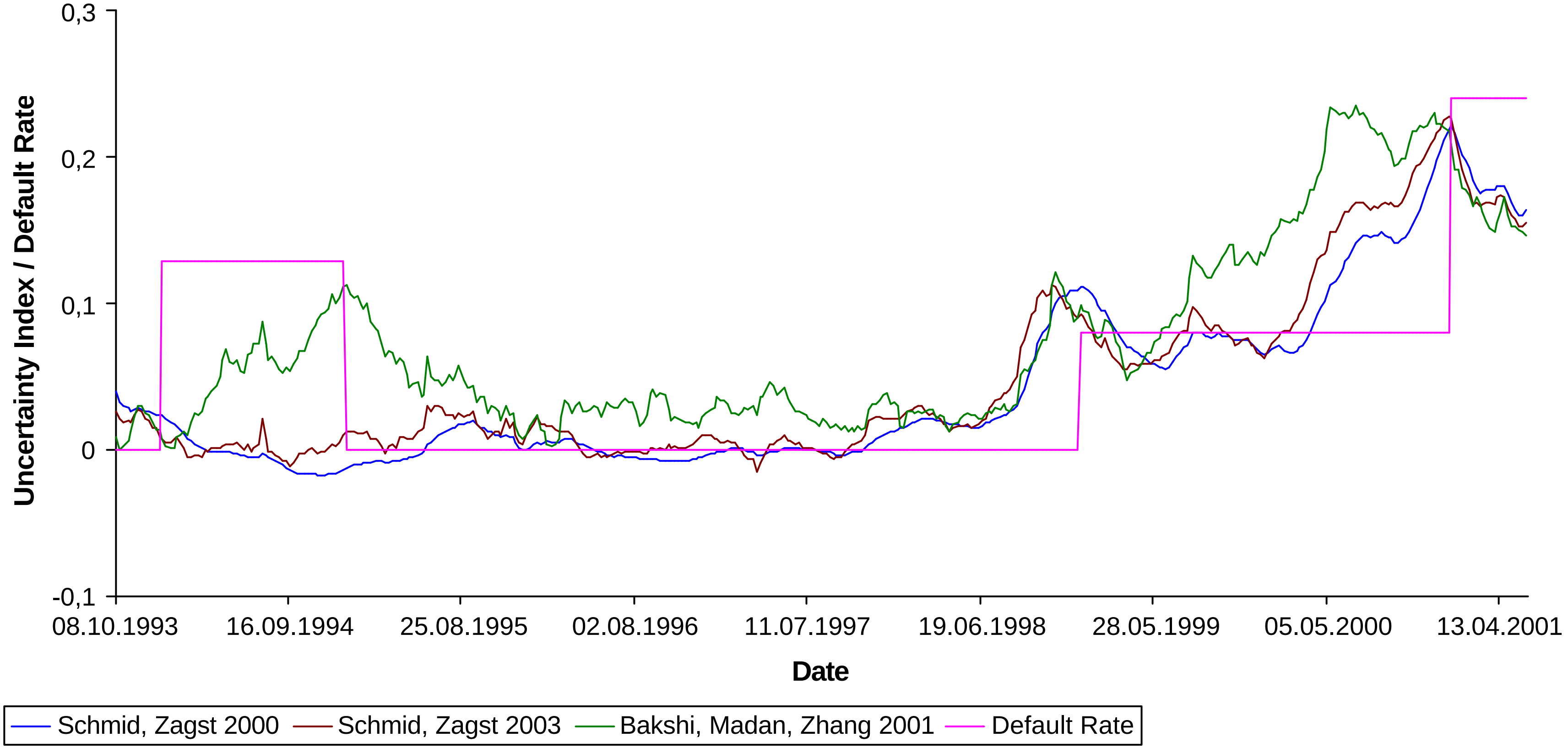
Average absolute Deviation of yields / spreads	Schmid, Zagst [2000]	Bakshi, Madan, Zhang [2001]	Schmid, Zagst [2003]
Treasury Strips	0.18687 (0.84627)	0.09754 (0.30299)	0.18686 (0.84619)
Industrials A2	0.09970 (0.10133)	0.08787 (0.20575)	0.06186 (0.09022)
Industrials BBB1	0.08742 (0.14296)	0.10503 (0.24667)	0.08089 (0.16488)

Average R^2	Schmid, Zagst [2000]	Bakshi, Madan, Zhang [2001]	Schmid, Zagst [2003]
Treasury Strips	0.9005 (0.8806)	0.9461 (0.8981)	0.8280 (0.8800)
Industrials A2	0.3922 (0.3320)	0.3083 (0.2676)	0.5308 (0.4591)
Industrials BBB1	0.4736 (0.4882)	0.2778 (0.2960)	0.4277 (0.4324)

* Values without brackets are in sample, values in brackets are out of sample

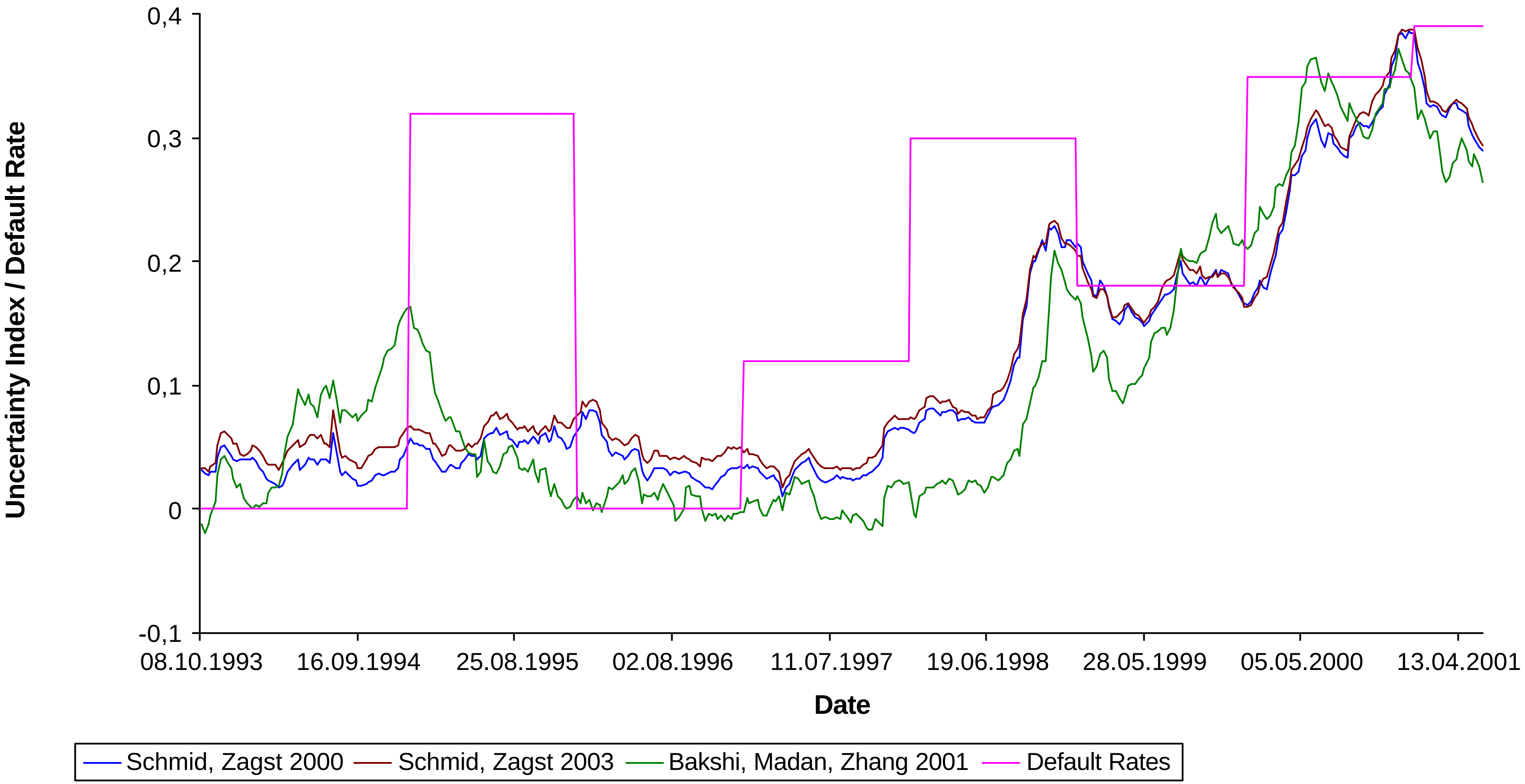
Model Comparison

Uncertainty Index vs. Default Probability for Rating A2



Model Comparison

Uncertainty Index vs. Default Probability for Rating BBB1



Generalization of the SZ-Model for the Pricing of Defaultable Bonds
Overview

- **Lessons from the Market**
- **Three Models for the Pricing of Defaultable Bonds**
- **Model Comparison**
- **Further Research**
  - Inclusion of Other Models
  - Integration of Other Macroeconomic Factors