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Private Research Institute for Financial Studies

Portfolio Optimization Under Credit Risk

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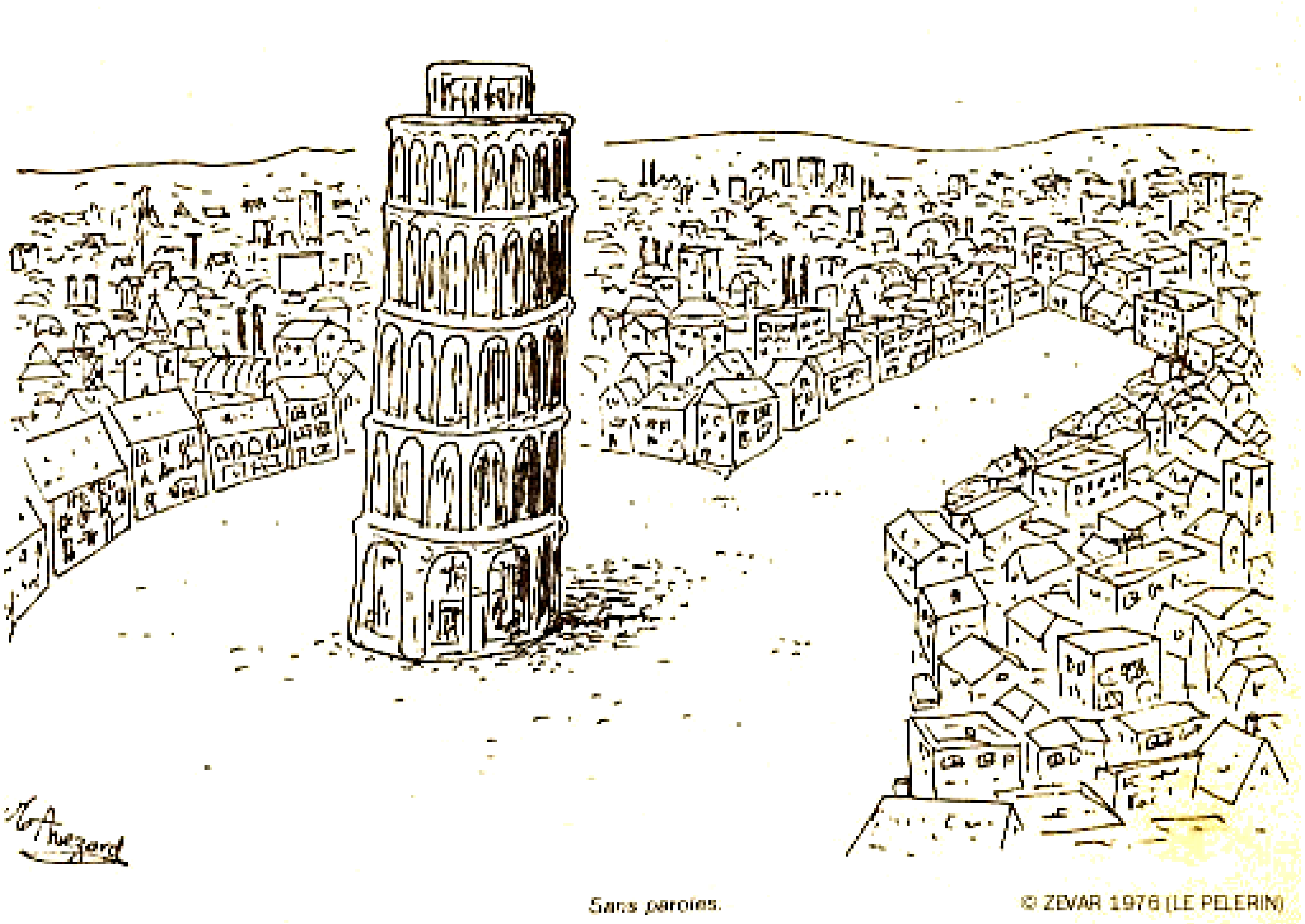
Madrid, October 18, 2001

Lessons from Pisa



Lessons from the Market

- **Credit Information**
  - ➢ Definition of Credit Risk
  - ➢ Yield Curve Behaviour
  - ➢ Quality Changes
  - ➢ Credit Spread Behaviour

Credit Risk
Definition according to the Dictionary of Financial Risk Management [1996]

Credit Risk is

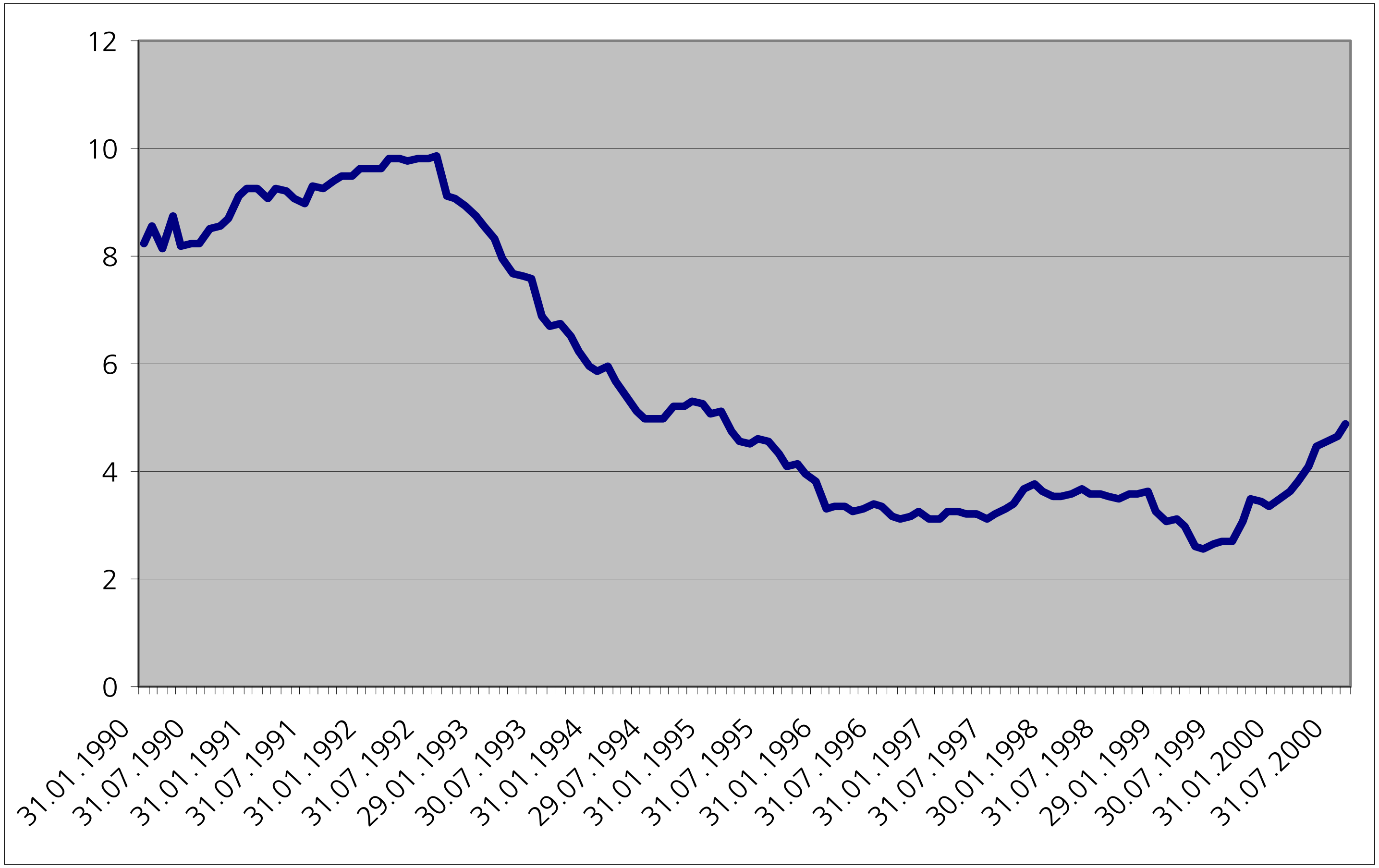
- *the exposure to loss* as a result of
  - default on a swap debt or other counterparty instrument
  - a decline in market value stemming from a credit downgrade of an issuer or counterparty
- *a component of return variability* resulting from the possibility of an event of default
- *a change in the market's perception of the probability of an event of default*, which affected the spread between two rates or reference indexes

and therefore is considered as

- a *classical downside risk*, i.e. the best case is a full repayment of the debt
- *but* may also include an *upside potential*, i.e. an increase in market value stemming from a credit upgrade of an issuer or counterparty

Interest Rates Do Exhibit a Mean Reverting Behaviour

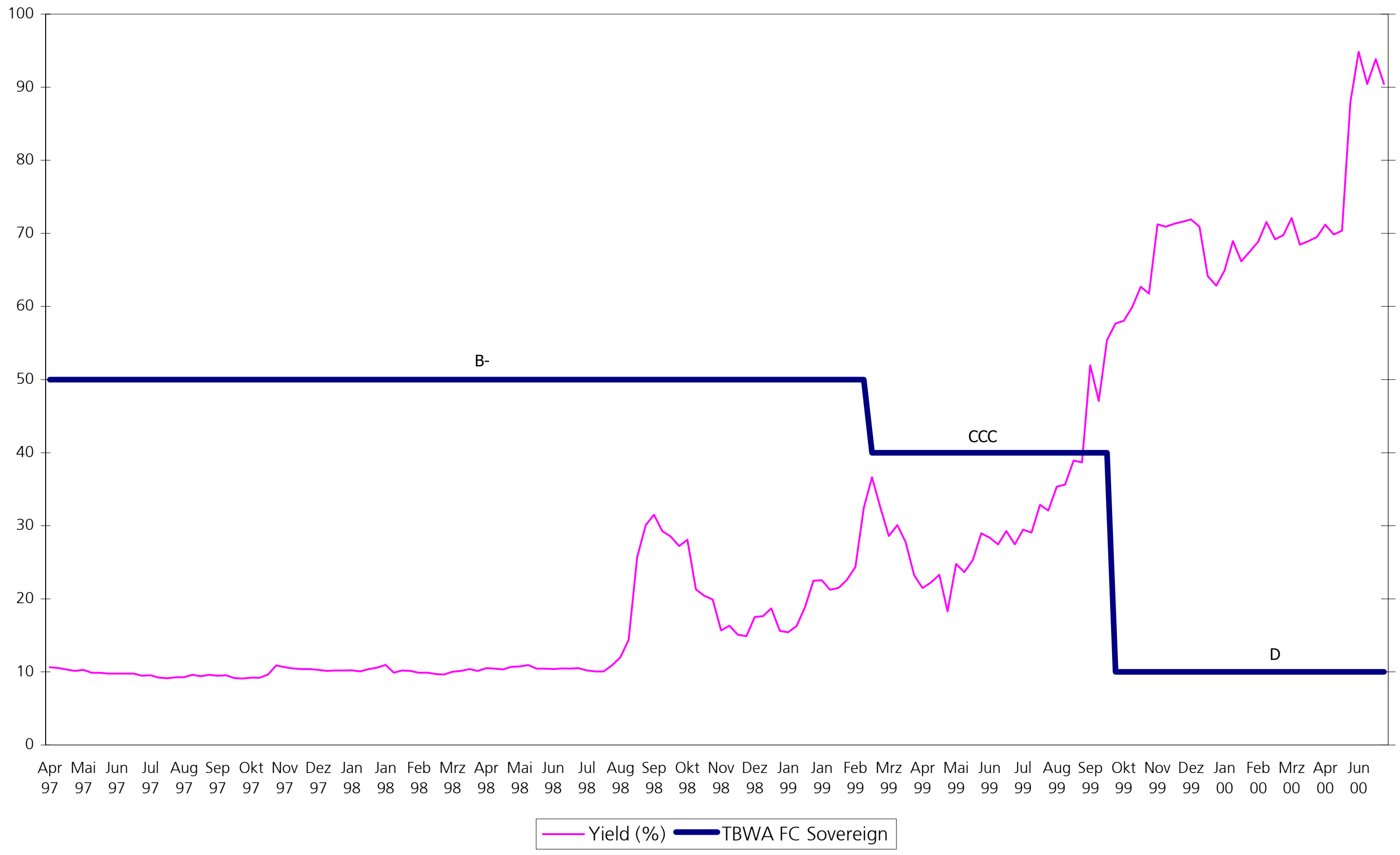
3-Monats Libor from 1990 until 2000 (in %)



Source: Bloomberg

The Quality Process is Continuous

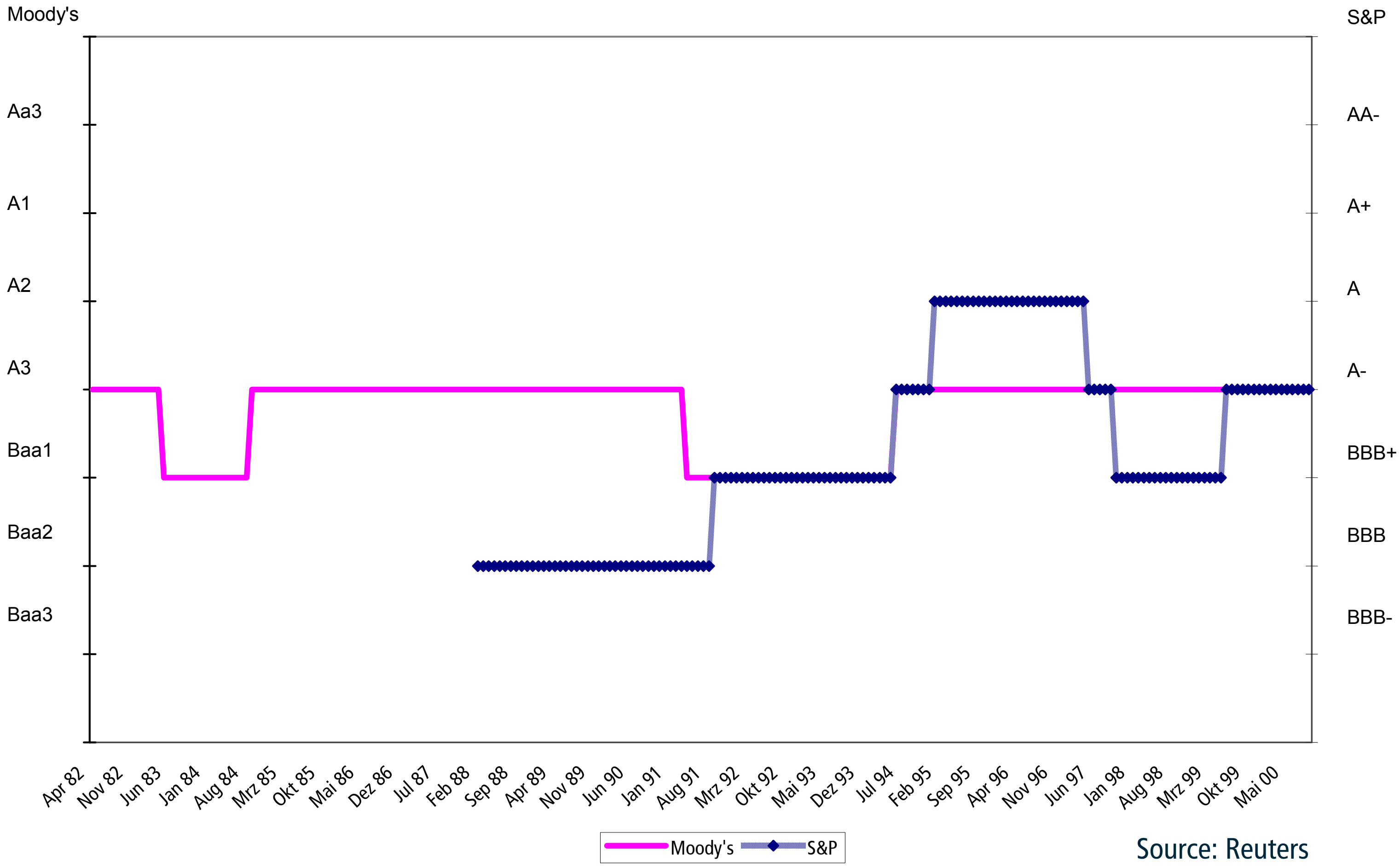
Ecuadorian Sovereign Bond BEEC 11.25% 25/04/2002 DEM



Source: Reuters

The Quality Process Does Exhibit a Mean Reverting Behaviour

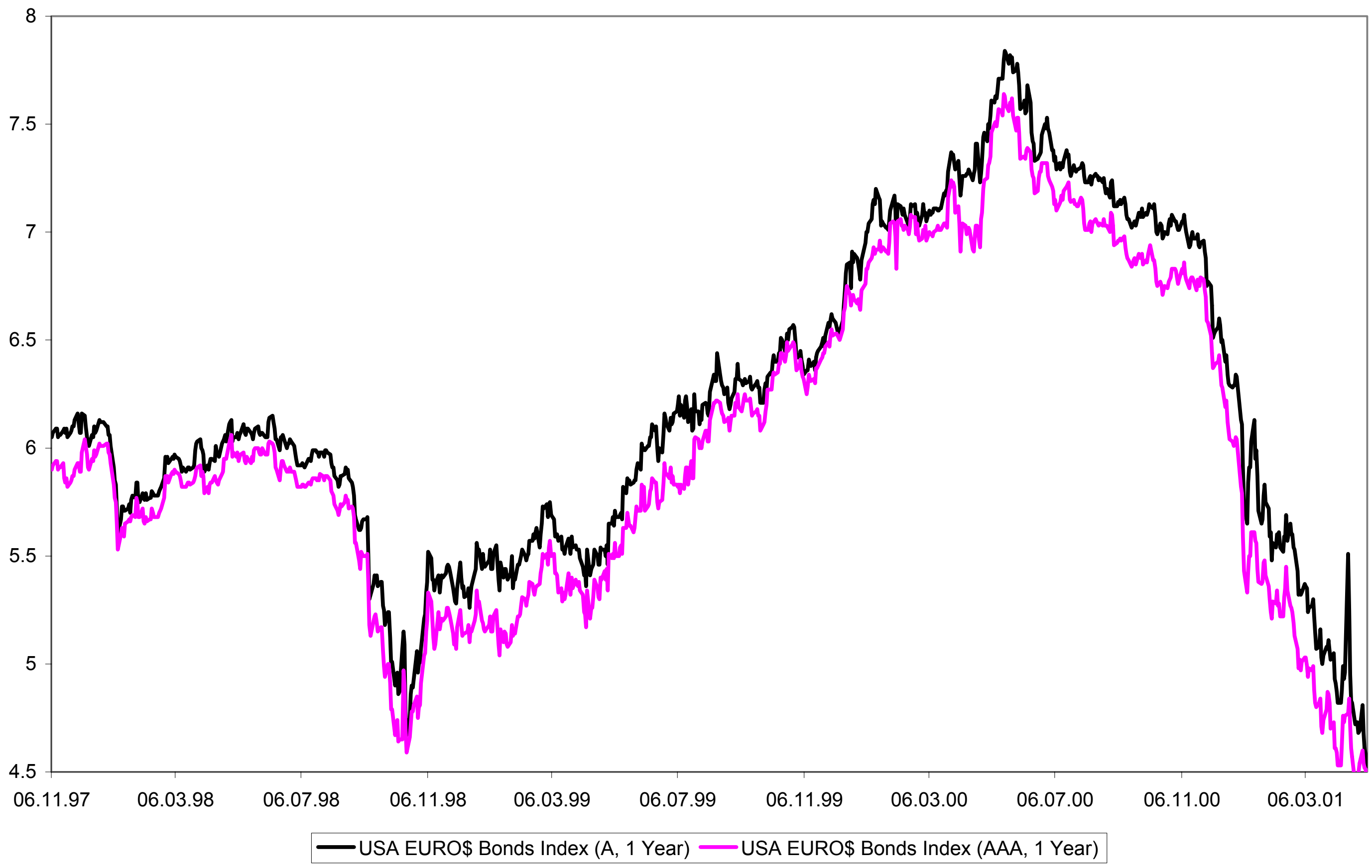
Rating History of Hertz from April 1982 until August 2000



Source: Reuters

The Extend of the Credit Spread Does Depend on the Rating

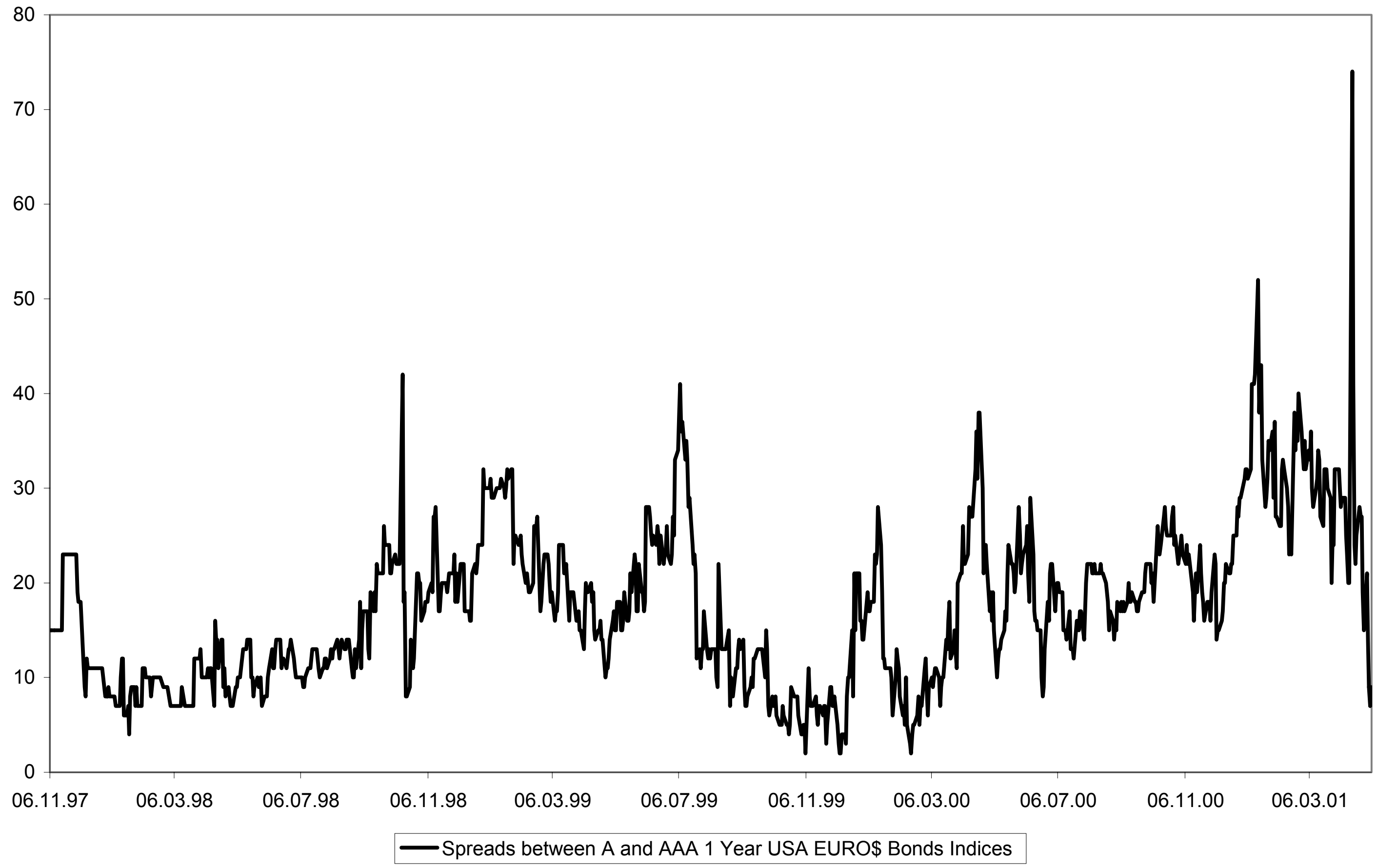
Yields (in %) of 1 Year Euro$ Bond Indices for the Ratings A and AAA



Source: Bloomberg

Credit Spreads Do Exhibit a Mean Reverting Behaviour

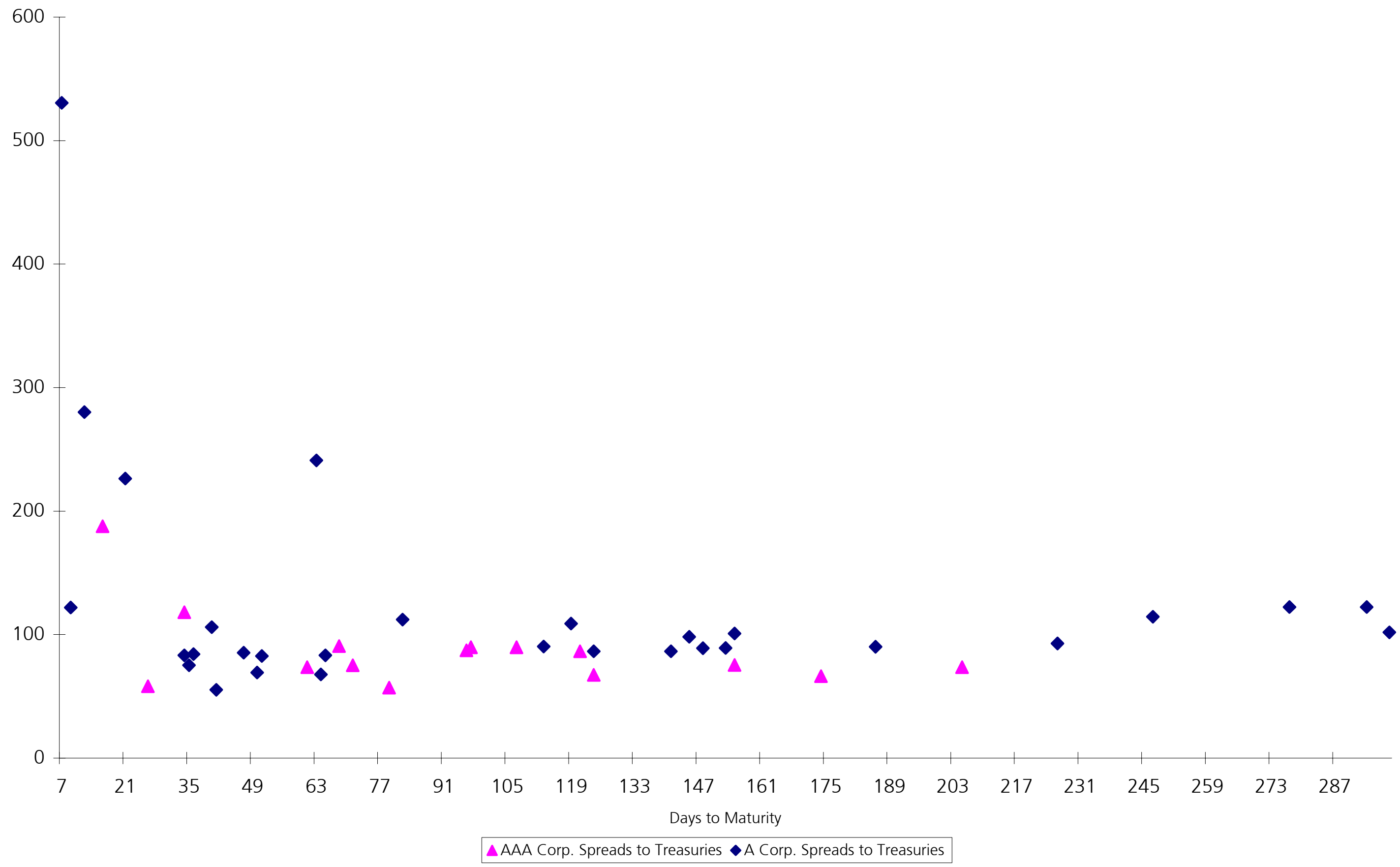
1 Year Credit Spread between A and AAA Euro$ Bond Indices (in bp)



Source: Bloomberg

Even Short-Term Credit Spreads May Be Positive

Credit Spreads of A and AAA Rated Corporate Bonds (bp) for Different Maturities



Source: Reuters

Models for the Pricing of Defaultable Bonds
Overview

- **Fundamental Models**

  Merton [1974], Black und Cox [1976],
  Longstaff und Schwartz [1995] etc.

- **Relative Models**

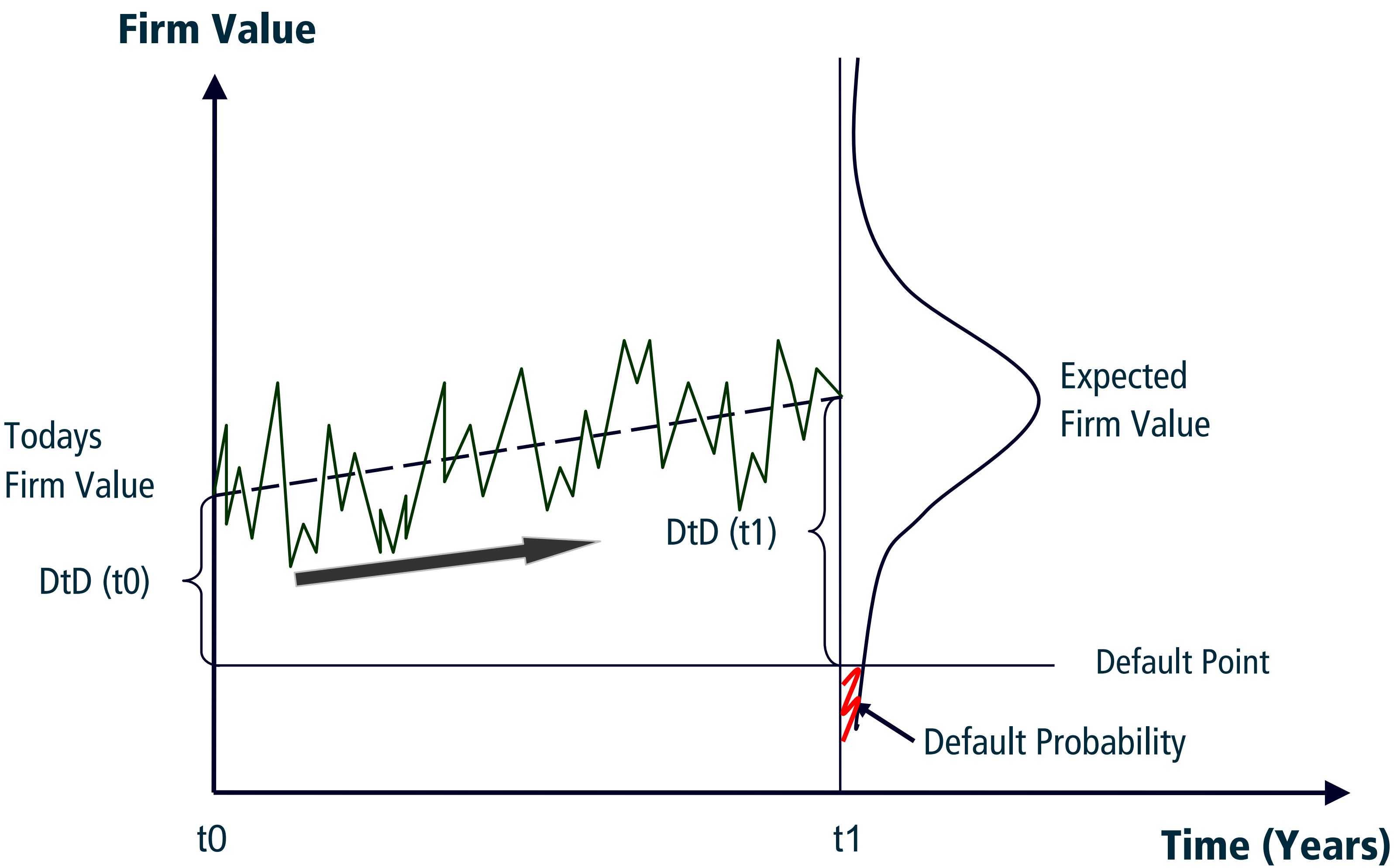
  Jarrow und Turnbull [1995],
  Madan und Unal [1994,1998],
  Duffie und Singleton [1994-1999] etc.

- **Hybride Models**

  Cathcart und El-Jahel [1998],
  Schmid und Zagst [2000] etc.

Fundamental Models

Modeling of the Default Probability



Fundamental Models
The Merton Model (1974)

Modeling of the non-defaultable short rate r,
the firm value V, and the default boundary D,
i.e. default probability and recovery rate are implicitely given

- Firm finances its activities via stocks and discount bonds at a notional D and maturity T
- Firm value V changes according to the SDE

$$dV_t = rV_t dt + \sigma V_t dW_t$$

with a standardized Brownian motion W and constant interest rate r and volatility σ

- At maturity T the creditors receive $\min(V_T, D) = D - \max(0, D - V_T)$

Hence, the value of the defaultable bond is given by

$$P^d(r, V, t, T) = D \cdot P(r, t, T) - Put(V, t, D, \sigma, r, T - t),$$

where *Put* is the Black-Scholes value of a European put option

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Relative Models
Modeling of the Default Probability

Exogenious modeling of the non-defaultable short rate r, the default probability q under the equivalent martingale measure, and the recovery rate y

Jarrow/Turnbull [1995]:

Default = first jump of a Poisson process with constant intensity parameter (hazard rate) h>0, i.e. the probability of a default in the interval (t,T] is given by

$$q(t,T) = 1 - \exp[-h \cdot (T - t)]$$

Madan/Unal [1998]:

The intensity parameter is modeled by a stochastic process $(h(t))_{t \geq 0}$, i.e. the probability of a default in the interval (t,T] is given by

$$q(t,T) = 1 - E_Q\left[\exp\left(-\int_t^T h(l)dl\right) | \Im_t\right]$$

Relative Models
Pricing Defaultable Bonds

The value of a defaultable discount bond at time t maturing at time T≥t is given by

$$P^d(r,h,t,T) = E_Q\left(\frac{B(t)}{B(T)} \cdot X(T) \mid \Im_t \right)$$

where

- Q is the martingale measure on the complete filtered probability space $\left(\Omega, \Im, Q, (\Im_t)_{t\in[0,T^*]}\right)$
- B(t) is the value of the money market account at time t
- X(T) is the value of the payoff of the bond at time T

Under appropriate independence assumptions we get:

$$\begin{aligned} P^d(r,h,t,T) &= E_Q\left[\frac{B(t)}{B(T)} \cdot X(T) \mid \Im_t \right] = P(r,t,T) \cdot E_Q\left[X(T) \mid \Im_t\right] \\ &= P(r,t,T) \cdot \left[1 - q(t,T) + y \cdot q(t,T)\right] \end{aligned}$$

where y denotes the recovery rate at time T

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The Model of Schmid and Zagst [2000]
Modeling of the Stochastic Processes under the Martingale Measure Q

- **Dynamic of the yield curve (non-defaultable short rate)**

$$dr(t) = [\theta_r(t) - \hat{a}_r \cdot r(t)]dt + \sigma_r d\hat{W}_r(t), t \in [0, T^*], \hat{a}_r > 0, \sigma_r > 0$$

- **Dynamic of the counterparty quality (uncertainty index)**

$$du(t) = [\theta_u - \hat{a}_u \cdot u(t)]dt + \sigma_u \sqrt{u(t)} d\hat{W}_u(t), t \in [0, T^*], \ \theta_u \geq 0, \hat{a}_u > 0, \sigma_u > 0$$

- **Dynamic of the yield spreads (short rate spreads)**

$$ds(t) = [b_s \cdot u(t) - \hat{a}_s \cdot s(t)]dt + \sigma_s \sqrt{s(t)} d\hat{W}_s(t), t \in [0, T^*], \ b_s > 0, \hat{a}_s > 0, \sigma_s > 0$$

- **The Wiener processes $\hat{W}_r, \hat{W}_u$, and $\hat{W}_s$ are uncorrelated***

* assumption can be relaxed

The Model of Schmid and Zagst [2000]
Evaluation of Non-Defaultable Bonds

Theorem 1 (Hull and White [1990]).

The time t value P(t,T) of a non-defaultable discount bond with maturity T≥t is given by

$$P(t,T) = A(t,T) \cdot e^{-B(t,T) \cdot r}$$

with

$$B(t,T) = \frac{1}{\hat{a}_r} \cdot \left(1 - e^{-\hat{a}_r \cdot (T-t)}\right)$$

and

$$\ln A(t,T) = \ln\left(\frac{P(0,T)}{P(0,t)}\right) - B(t,T) \cdot \frac{\partial \ln P(0,t)}{\partial t} - \frac{\sigma_r^2}{4\hat{a}_r^3} \cdot \left(e^{-\hat{a}_r \cdot T} - e^{-\hat{a}_r \cdot t}\right)^2 \cdot \left(e^{2 \cdot \hat{a}_r \cdot t} - 1\right).$$

The Model of Schmid and Zagst [2000]
Evaluation of Defaultable Bonds

Theorem 2 (Defaultable Discount Bond, Schmid und Zagst [2000]).

The time t value $P^d(t,T)$ of a defaultable discount bond with maturity $T \geq t$ is given by

$$P^d(t,T) = A^d(t,T) \cdot e^{-B(t,T)\cdot r - C(t,T)\cdot s - D(t,T)\cdot u}$$

with

$$C(t,T) = \frac{1 - e^{-\delta_s \cdot (T-t)}}{K_1^{(s)} - K_2^{(s)} \cdot e^{-\delta_s \cdot (T-t)}} \quad \text{with} \quad \delta_s = \sqrt{\hat{a}_s^2 + 2 \cdot \sigma_s^2}$$

$$\text{and} \quad K_k^{(s)} = \frac{1}{2} \cdot \left(\hat{a}_s - (-1)^k \cdot \delta_s\right),\ k \in \{1,2\}$$

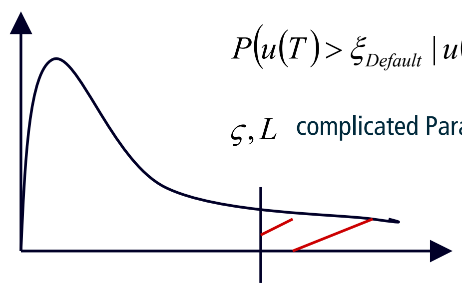
$$D(t,T) = -\frac{2 \cdot v'(t,T)}{\sigma_u^2 \cdot v(t,T)} \quad \text{with} \quad v(t,T) \text{ complicated}$$

$$A^d(t,T) = A(t,T) \cdot e^{-S(t,T)} \quad \text{with} \quad S(t,T) = -\frac{2 \cdot \theta_u}{\sigma_u^2} \cdot \ln\left|\frac{v(T,T)}{v(t,T)}\right|$$

Hybride Models as Generalization of Fundamental and Relative Models

Fundamental Models:

- Potential consideration of structural information (firm value)
- Makroeconomic factors as explaining factors for the quality of the debtor
- Definition of default



$$P\left(u(T) > \xi_{Default} \mid u(t)\right) = 1 - F_{\frac{4\cdot\theta_u}{\sigma_u^2},\varsigma}\left(\frac{\xi_{Default}}{L}\right)$$

ς, L complicated Parameters, dependent on Model Parameters

Relative Models:

- Short rate credit spread can be expressed using the intensity and recovery rate process
- $q(t,T) = 1 - E_Q\left[\exp\left(-\int_t^T h(l)\,dl\right) \mid \Im_t\right]$ vs. $q(t,T) = 1 - F_{\frac{4\cdot\theta_u}{\sigma_u^2},\varsigma}\left(\frac{\xi_{Default}}{L}\right)$

Model Comparison
Fundamental, Relative, and Hybride Models

	Fundamental Models	Relative Models	Hybride Models
Complexity	--	+	+
Market- and Structural Information	+	-	++
Model Performance	-	+	+
Consideration of Ratings	-	++	++
Pricing of Credit Derivatives	--	++	++

The Model of Schmid and Zagst [2000]

Results and Applications

- Cox-Ingersoll-Ross model to describe the short rate
- Numerical procedure for the pricing of credit derivatives allowing for correlations of the Wiener processes
- *Empirical parameter estimation using yield and spread curve data*
- Implicite parameter estimation using cap prices, yield curve, and rating information
- *Simulation, risk and portfolio management*

The Model of Schmid and Zagst [2000]

Empirical Parameter Estimation

- 33 AAA-rated German discount, 36 AA-rated Italian discount, and 38 (A-)-rated Greek coupon government bonds
- Maturities of the bonds between 0.36 and 20.8 years
- 22 month time series of Reuters daily bond prices from November 1, 1999 until October 23, 2000
- All prices are denominated in Euros
- German bonds are considered to be free of default risk
- Parameter estimation using the Kalman filter

The Model of Schmid and Zagst [2000]

Parameter Estimates

Parameter	Estimation Germany
θ_r (%)	1.4413
a_r	0.2382
σ_r (%)	1.5581
λ_r	-0.0861
r(0) (%)	4.2434

$$\hat{a}_x = a_x + \lambda_x \cdot \sigma_x^2, \;\; x \in \{r, u, s\}$$

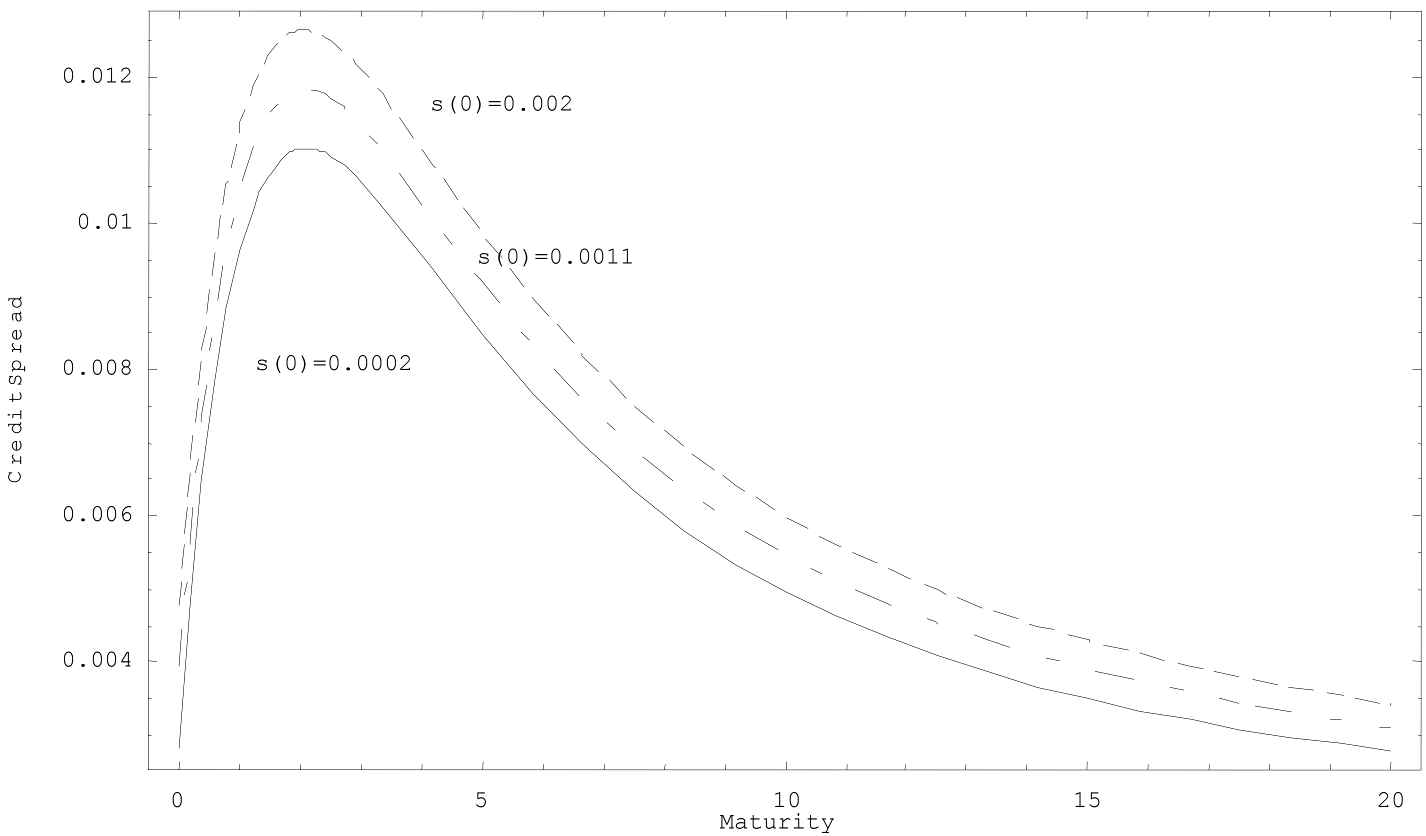
Parameter	Estimation Italy	Estimation Greece
b_s	0.2748	0.3432
a_s	0.0477	0.1678
σ_s (%)	15.832	44.6885
λ_s	-1.8987	-0.2159
s(0) (%)	0.1296	0.2325
θ_u (%)	0.0031	0.0392
a_u	0.0687	0.0750
σ_u (%)	3.0482	6.7231
λ_u	-1.2281	-1.1091
u(0) (%)	0.5112	1.2704

The Model of Schmid and Zagst [2000]
Statistical Tests

- ✓ Test on *Normal distribution* of the standardized difference between the new observations and the corresponding Kalman filter forecasts (standardized Kalman filter innovations) as proposed in Greene [1993], p. 310 (see Harvey [1989], p. 260, for a similar test)
- ✓ Test if the filtered values for the Wiener processes are *uncorrelated* as proposed in Larsen and Morris [2001], p. 626
- ✓ Box-Ljung tests on *serial correlation* of the standardized Kalman filter innovations (Q-statistic)
- ✓ Tests on *homoscedasticity* of the standardized Kalman filter innovations (H-statistic)
- ✓ *In-sample* tests on the *model performance* (linear regression)
- ✓ *Out-of-sample* tests on the *model performance* (linear regression)

The Model of Schmid and Zagst [2000]

Illustrations: Structure of the Model Implied Credit Spreads



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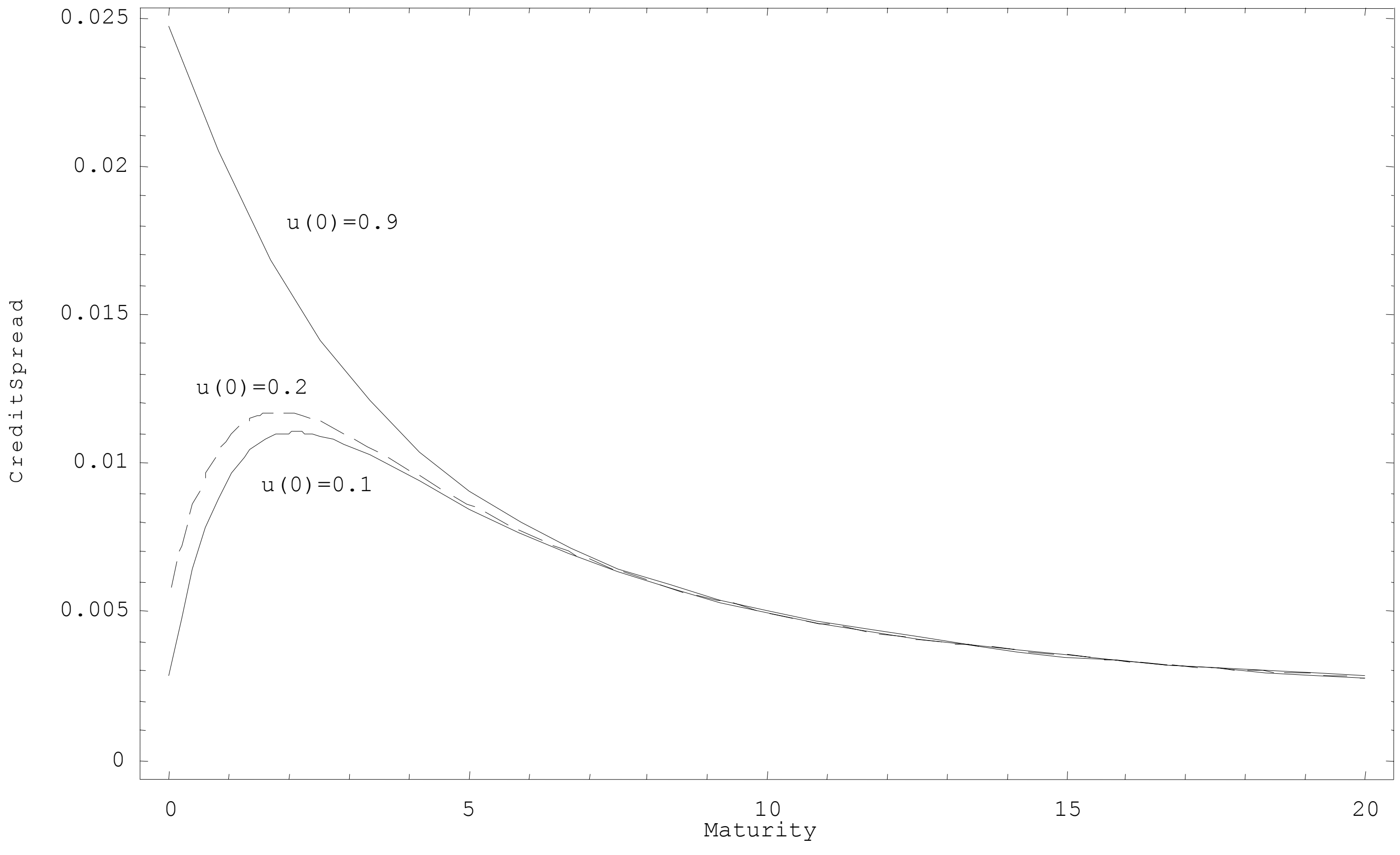


Portfolio Optimization under Shortfall and Liability Constraints

- **Credit Portfolio Management**
  - ➢ Definitions
  - ➢ Simulation
  - ➢ Downside Risk
  - ➢ Case Study

Credit Portfolio Management
Definitions

- Planning horizon $T \in [0,T^*]$
- $P_1(t),P_2(t),\ldots,P_n(t)$ prices of sovereign bonds $i=1,\ldots,n$ at time $t \in [0,T]$
- $\varphi=(\varphi_1,\varphi_2,\ldots,\varphi_n)$ portfolio of the sovereign bonds $i=1,\ldots,n$
- $B(t)$ benchmark value at time $t \in [0,T]$
- $\mathbb{T}=\{T_1,T_2,\ldots,T_L\} \subset [0,T]$ liability payment dates
- $L(t)$ liability payment at time $t \in \mathbb{T}$
- Coupon payments bewteen the payment dates are put on a cash account

Credit Portfolio Management
Simulation

- $F=(F_1,F_2,\ldots,F_m)=(r,s^{(I)},u^{(I)},s^{(G)},u^{(G)})$ vector of risk factors with simulations F^k and probability $p_k>0$, $k=1,\ldots,K$
- $V_i(t)=V_i(F,t)$, $i=1,\ldots,n$, value of the stochastic variable V_i (cash account or bond plus potential coupon payments between times 0 and t) at time $t\in[0,T]$ with simulations $V_i^k(t)=V_i(F^k,t)$, $k=1,\ldots,K$, $i=1,\ldots,n$
- (Simulated) future value $V^k(\varphi,t)$ of portfolio $\varphi=(\varphi_1,\varphi_2,\ldots,\varphi_n)$ at time $t\in[0,T]$:

$$V^k(\varphi,t)=\sum_{i=1}^{n}\varphi_i\cdot V_i^{(k)}(t),\ k=1,\ldots,K$$

Credit Portfolio Management
Downside Risk

- Lower Partial Moment (LPM) of order $l \in \mathbb{N}$ corresponding to the benchmark B(t):

$$LPM_l(\varphi, V, B, t) = \sum_{\substack{k=1 \\ V^k(\varphi,t) < B(t)}}^{n} p_k \cdot \left(B(t) - V^{(k)}(\varphi, t)\right)^l$$

- l=0 shortfall probability
- l=1 (expected) regret
- **Goal:** Maximize final expected portfolio value (return)

  under

  Allocation constraints, limited downside risk, covered liability stream

Credit Portfolio Management
Case Study

- Planning horizon 2 years (2Y)
- Liability payment dates $\mathcal{T}$={6M,1Y,1.5Y,2Y}
- Liability stream (50000,100000,200000,400000) Euro
- Budget = benchmark = 1 Mio. Euro
- No short sales
- Maturity of the bonds semiannual between 1 and 3 years
- Coupons of the German, Italian, and Greek bonds are 6%,7%, and 8%
- Coupon payments bewteen the payment dates are put on a cash account

Credit Portfolio Management

Example 1: No Shortfall and Liability Constraints

- Optimal portfolio:

  9,658 (1 Mio. Euro) of the 2-year 8% Greek bond

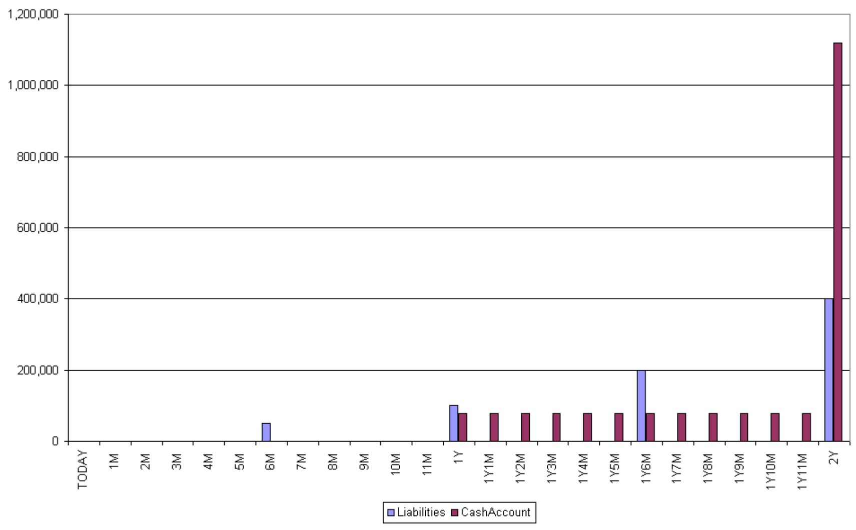
- Expected final value:

  1,120,337 Euro corresponding to an expected rate of return of 5.68%

- Probabilities of falling below the benchmark at times $t \in \mathbb{T}$:

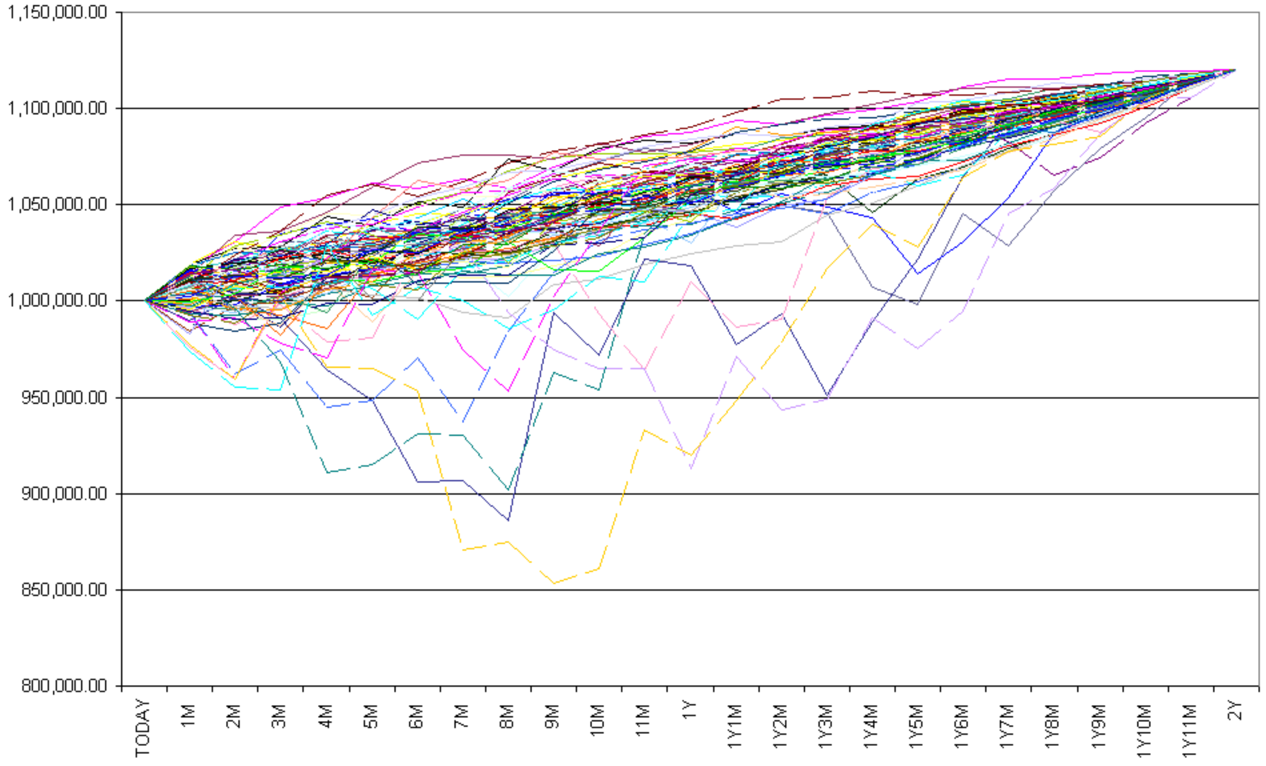
  (5%,2%,1%,0%)

- Liability stream is not covered by this portfolio:



Credit Portfolio Management

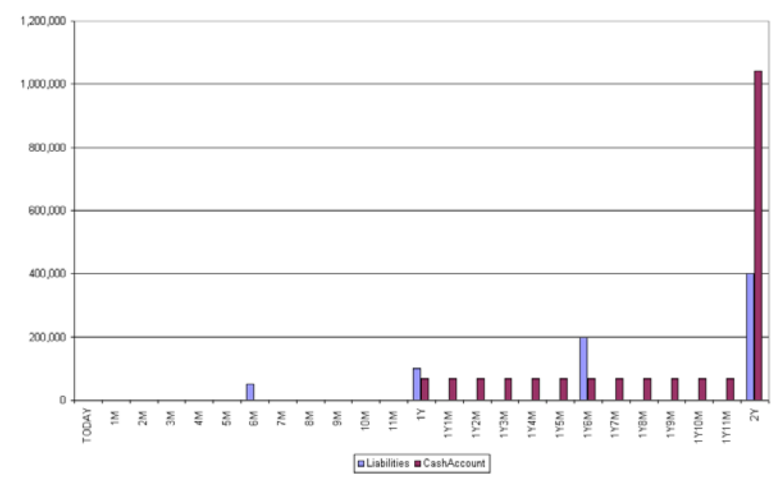
Example 1: Simulation of the Portfolio Value (including Coupon Payments)



Credit Portfolio Management

Example 2: Additional Shortfall Constraint, No Liability Constraint

- Shortfall probability 1%
- Optimal portfolio:
  3,008 (311,428 Euro) of the 2-year 8% Greek bond
  2,485 (253,287 Euro) of the 2-year 7% Italian bond
  3,604 (369,108 Euro) of the 2-year 6% German bond
  641 (66,177 Euro) of the 3-year 6% German bond
- Expected final value:
  1,105,841 Euro corresponding to an expected rate of return of 5.03%
- Probabilities of falling below the benchmark at times $t \in \mathbb{T}$: (1%,1%,0%,1%)
- Liability stream is not covered by this portfolio:



Credit Portfolio Management

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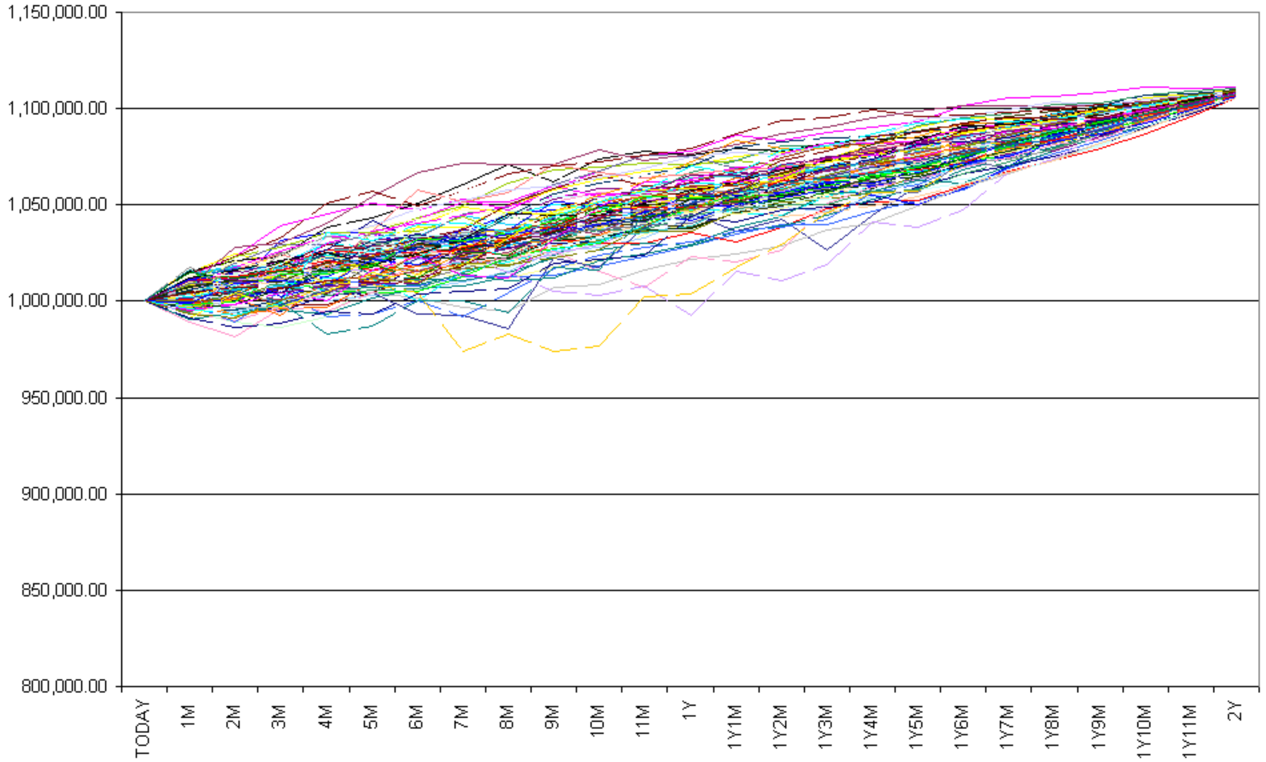


Credit Portfolio Management

Example 3: Additional Shortfall and Liability Constraint

- Shortfall and liability constraint at a probability of 1%
- Optimal portfolio:

  3,050 (315,815 Euro) of the 2-year 8% Greek bond

  2,133 (217,370 Euro) of the 2-year 7% Italian bond

  1,869 (196,040 Euro) of the 1.5-year 7% Italian bond

  465 (46,900 Euro) of the 1-year 7% Italian bond

  1,825 (186,959 Euro) of the 2-year 6% German bond

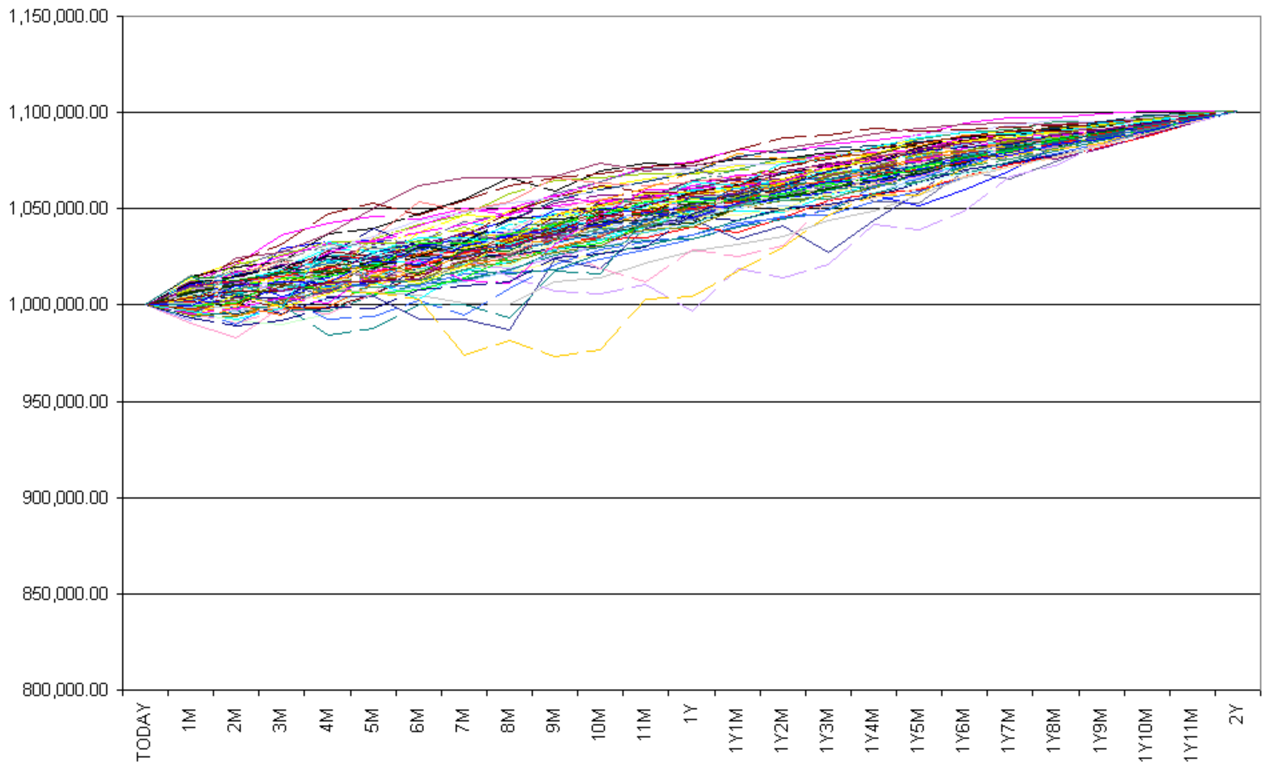
  36,916 Euro cash
- Expected final value:

  1,099,185 Euro corresponding to an expected rate of return of 4.73%
- Probabilities of falling below the benchmark

  at times $t \in \mathbb{T}$: (1%,1%,0%,0%)

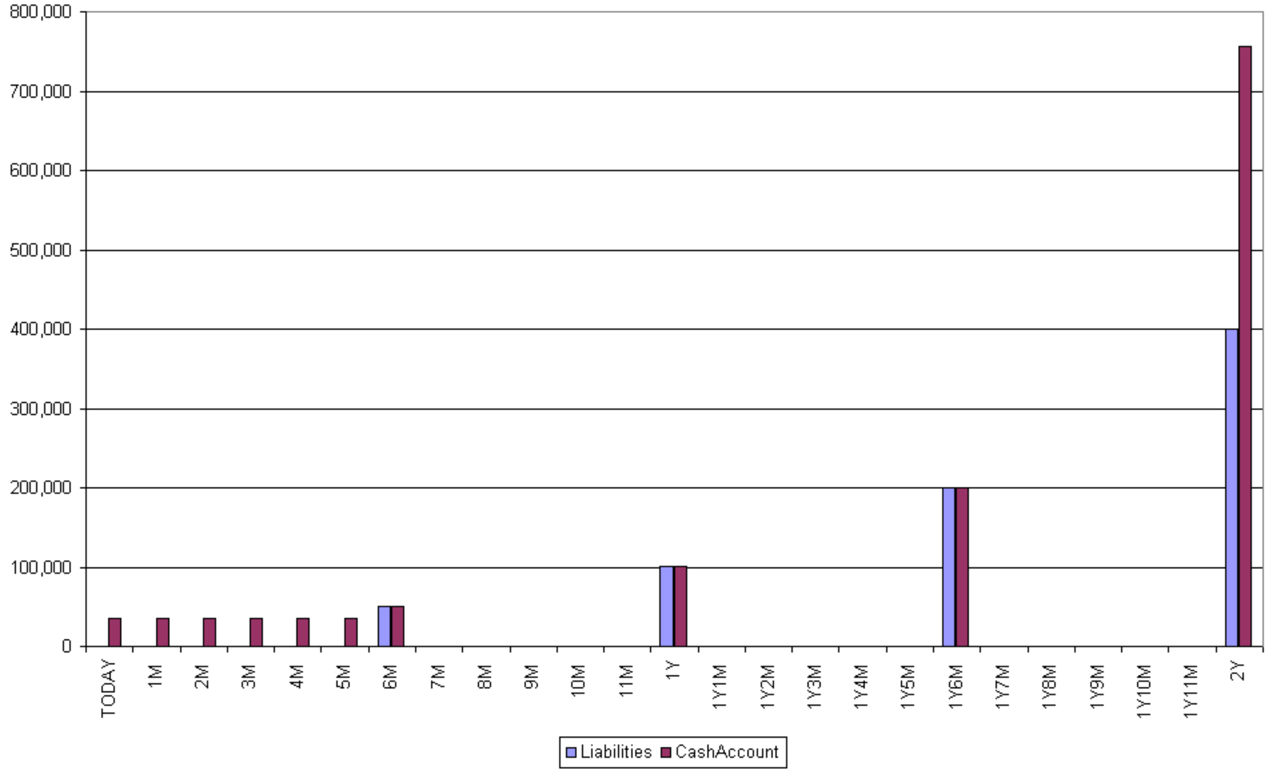
Credit Portfolio Management

Example 3: Simulation of the Portfolio Value (including Coupon Payments)



Credit Portfolio Management

Beispiel 3: Cash Account and Liabilities



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