Computational Tools for the Analysis of Market Risk

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Introduction

- Who measures **Risk**?
  - Financial institutions.
  - Supervisory agencies.
  - Rating agencies.
- Why measure **Risk**?
  - Control / manage risk associated with extreme events in the markets.
  - Capital adecuacy Regulations.

Introduction

- In theory: Markets are efficient
  - ➢Absence of arbitrage opportunities.
  - ➢No systematic trends.
  - ➢Very short term memory.
- Model: Black-Scholes
  - ➢Log of daily returns of an asset are distrubuted according to a normal distribution.
  - ➢Two parameters:
    - Risk free **interest rate**.
    - **Volatility**.

Financial Risks

- Which are the sources of **Risk**?
  - Market risks.
  - Credit risks.
  - Liquidity risks.
  - Operational risks.
  - Legal risks.

Measuring Risk

- What (precisely) do we mean by **Risk**?

  Uncertainty associated with unexpected fluctuations in the value of assets or liabilities of interest due to movements in financial markets.

- How do we measure **Risk**?
  - Value-at-Risk.
  - Shortfall.
  - MaxVaR.
  - Unconditional / Conditonal Measures

MatRisk

- **Matrisk** is an interactive tool to carry out Risk analysis
  - **Normal VaR**.
  - Parametric fit to **mixtures of normals**.
  - **Shortfall** analysis.
  - **MaxVaR** analysis**.**

- Entirely developed in MATLAB.
- Development in Java under way.

Value at Risk

- **Definition**: Estimate of worst loss in the value of a portfolio over a time horizon with a given probability.
  - <u>Time horizon</u> (holding period): Usually short (eg. 1 day)
  - <u>Probabiliy</u>: Usually high (eg. 99 %).

- VaR = Percentile of returns distribution at a given probability level .

Sample-derived percentiles.

- **Discreteness of extreme events**.
  - No out-of-sample predictions.
  - Influence of sample size.
  - Sensitivity to time window.
  - Highly variable estimations of VaR.

- These disadvantages can be (partially) alleviated by a parametric approach.

Parametric Fit

- **Likelihood function:**

$$L(x_1, x_2, \ldots, x_N) = \prod_{n=1}^{N} P(x_n)$$

- **Fit:** Maximization of Log-likelihood.
- **Hypothesis for the pdf:** Normal, mixture of normals, hyperbolic.

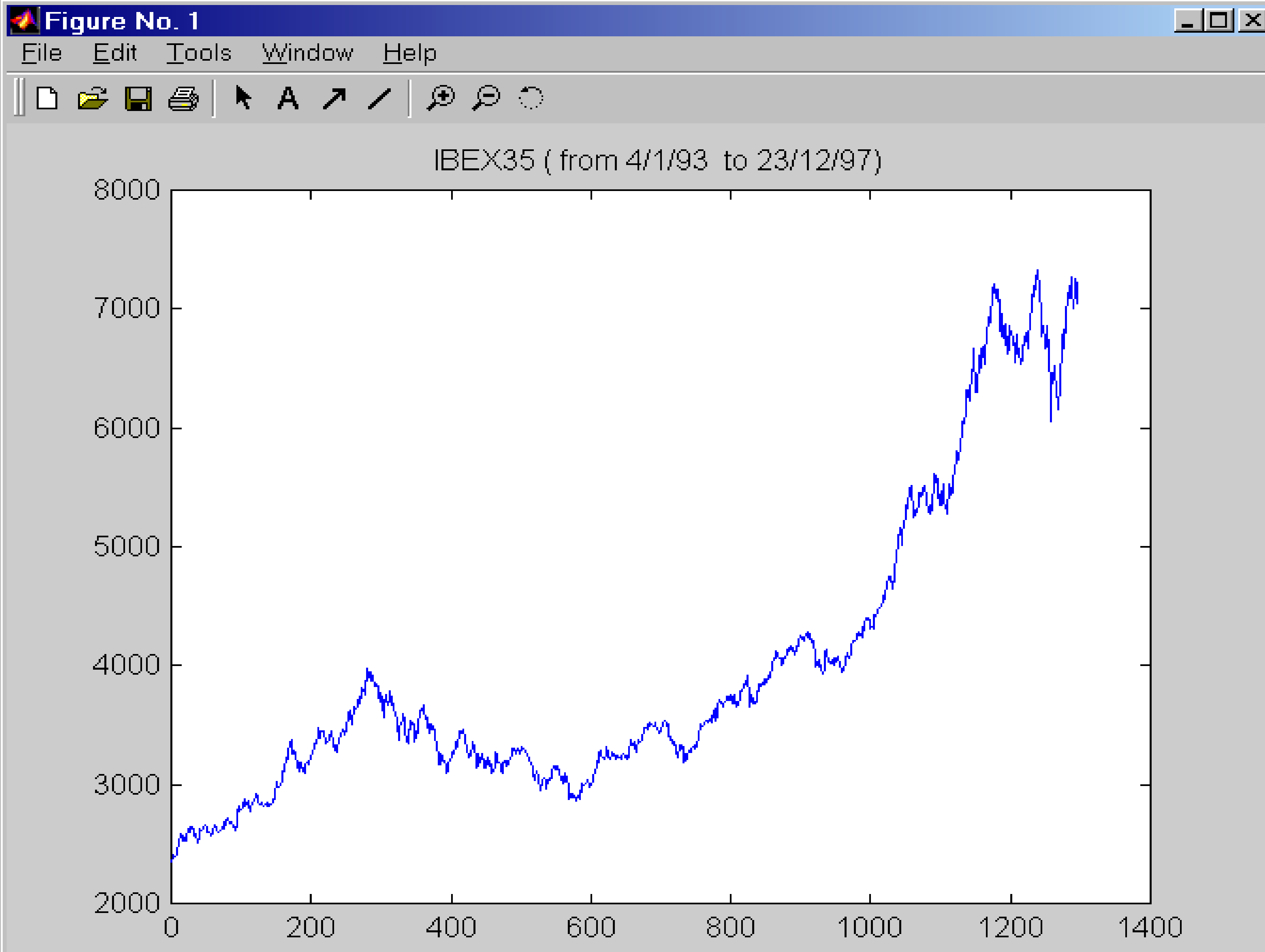


Figure No. 1
File Edit Tools Window Help
IBEX35 (from 4/1/93 to 23/12/97)
8000
7000
6000
5000
4000
3000
2000
0
200
400
600
800
1000
1200
1400

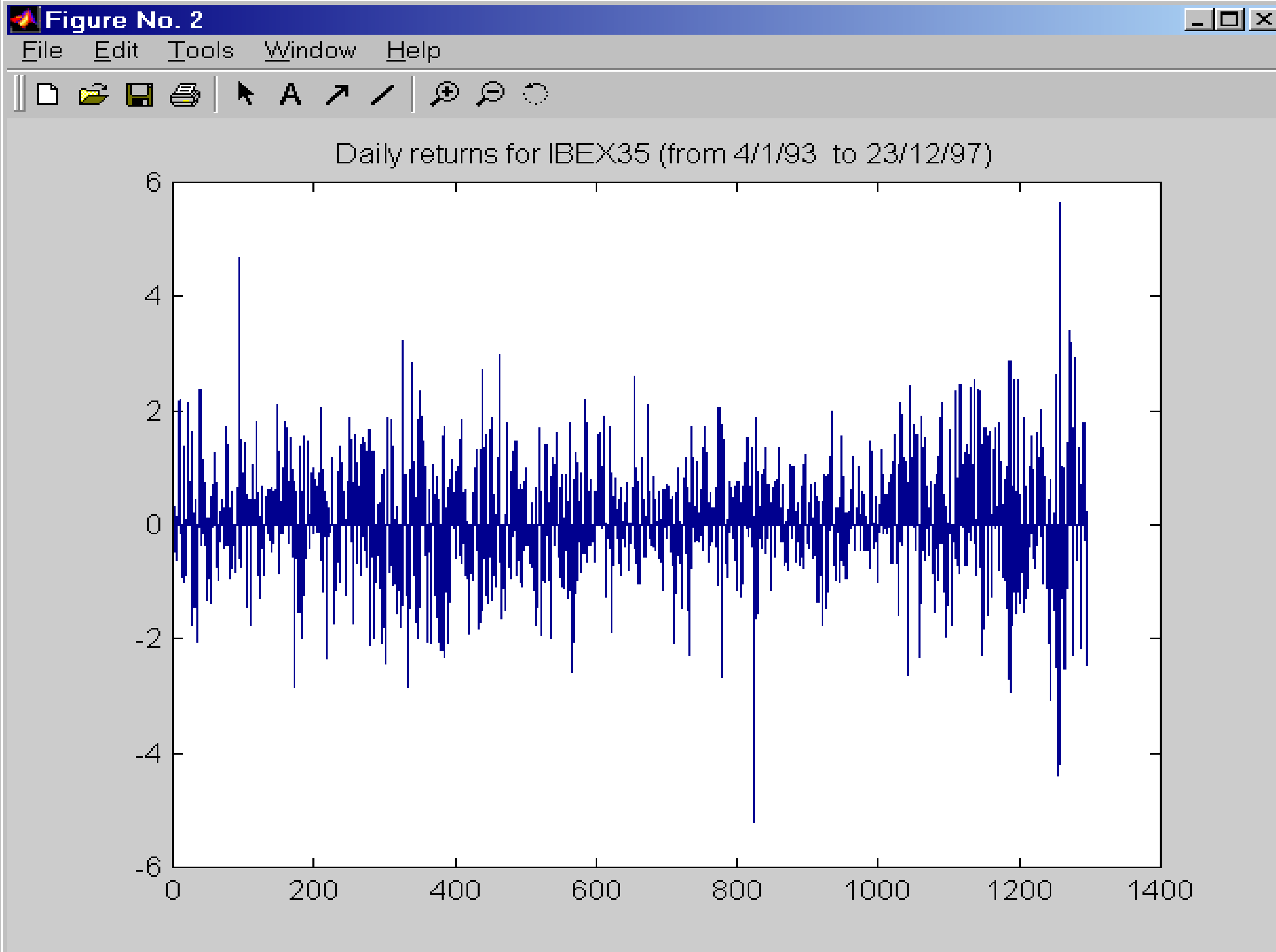


Figure No. 2
File Edit Tools Window Help
Daily returns for IBEX35 (from 4/1/93 to 23/12/97)
6
4
2
0
-2
-4
-6
0
200
400
600
800
1000
1200
1400

Normal model

- **Probability distribution**

$$N(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- **Problems:**
  - Heavy tails
  - Non-independence of returns

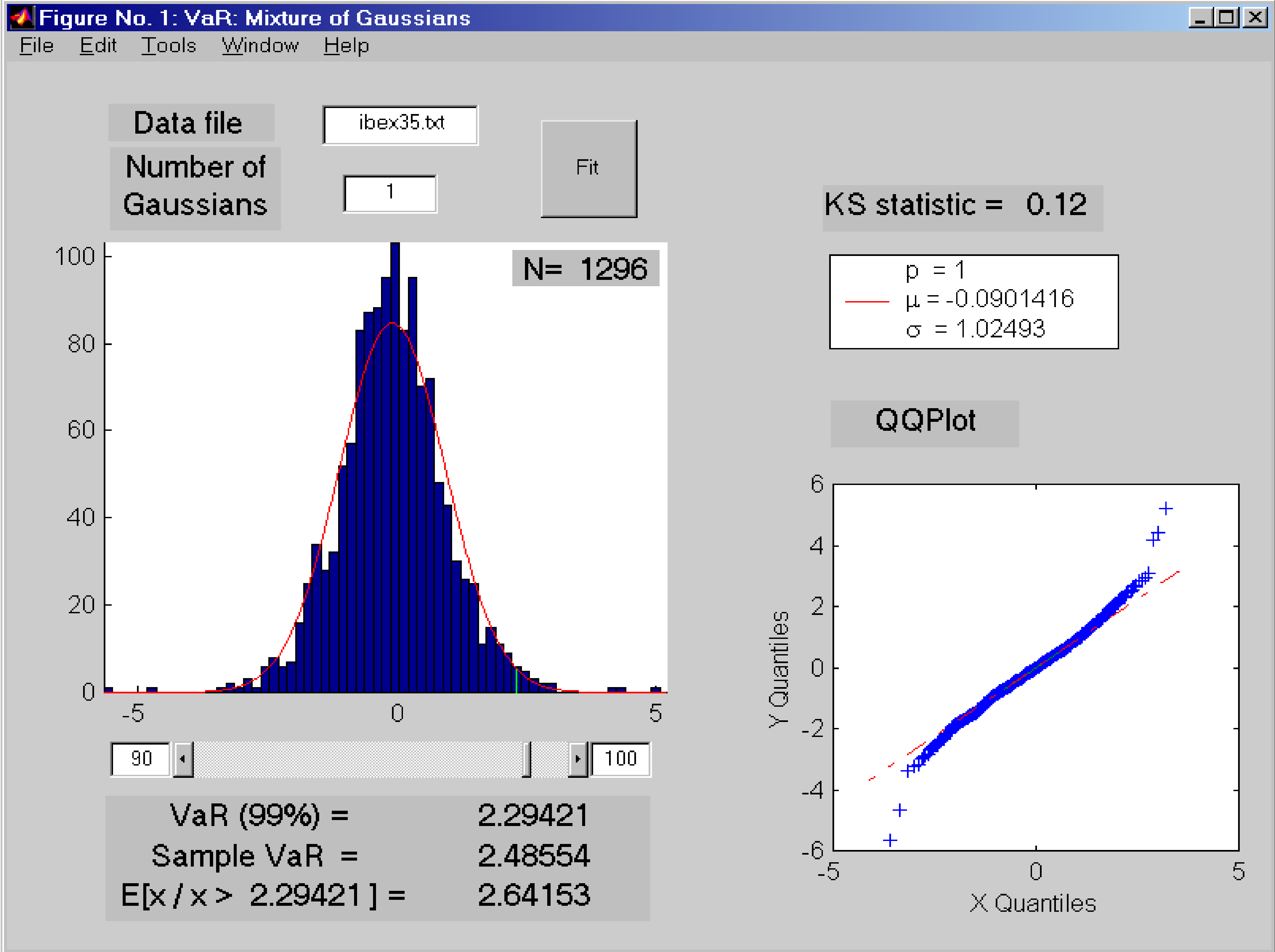


Figure No. 1: VaR: Mixture of Gaussians
File Edit Tools Window Help
Data file
ibex35.txt
Number of Gaussians
1
Fit
KS statistic = 0.12
p = 1
μ = -0.0901416
σ = 1.02493
QQPlot
N= 1296
90
100
VaR (99%) = 2.29421
Sample VaR = 2.48554
E[x / x > 2.29421] = 2.64153
Y Quantiles
X Quantiles

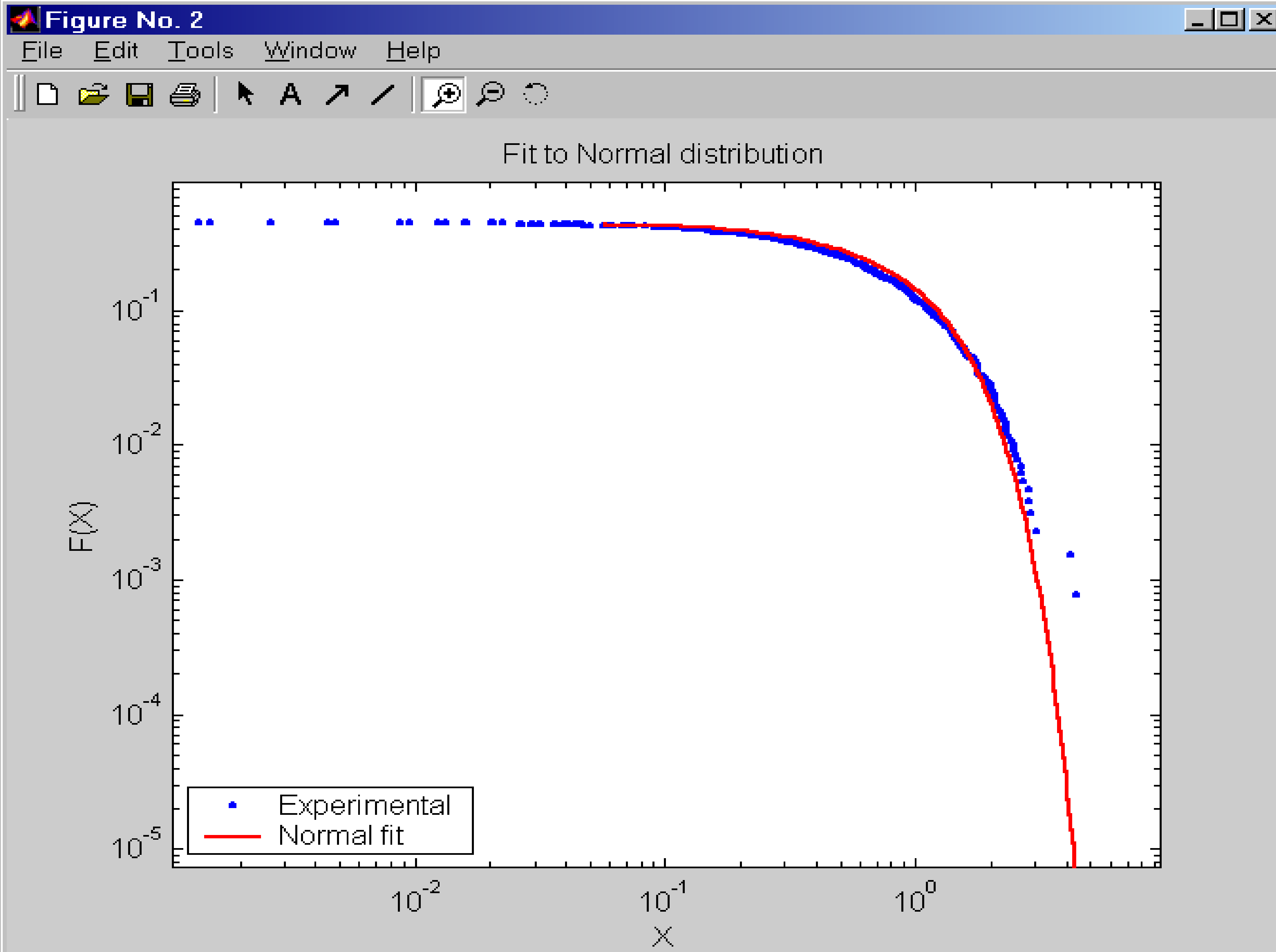


Figure No. 2
File
Edit
Tools
Window
Help
Fit to Normal distribution
F(X)
X
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{0}
Experimental
Normal fit

Mixture of Gaussians

- **Probability distribution**

$$P(x) = \sum_{k=1}^{K} p_k \, \mathrm{N}(x; \mu_k, \sigma_k)$$

- **Warning**: The log-likelihood function does not have a global minimum for mixtures of Gaussians.
- **Solution**: Use a modified log-likelihood function. (Hamilton 1991)

Mixture model

- X_T is taken from a mixture of normals

$$X_T = \begin{cases} N(\mu_1, \sigma_1) & \text{with probability } p \\ N(\mu_2, \sigma_2) & \text{with probability } (1-p) \end{cases}$$

  - Intuitively appealing: Close to BS world
  - Useful to capture extreme events (Ej. In VaR)
  - **But:**
    - No clear dynamical picture (a jump process?)
    - More parameters to fit (Overfitting?)
    - Tails are still exponential

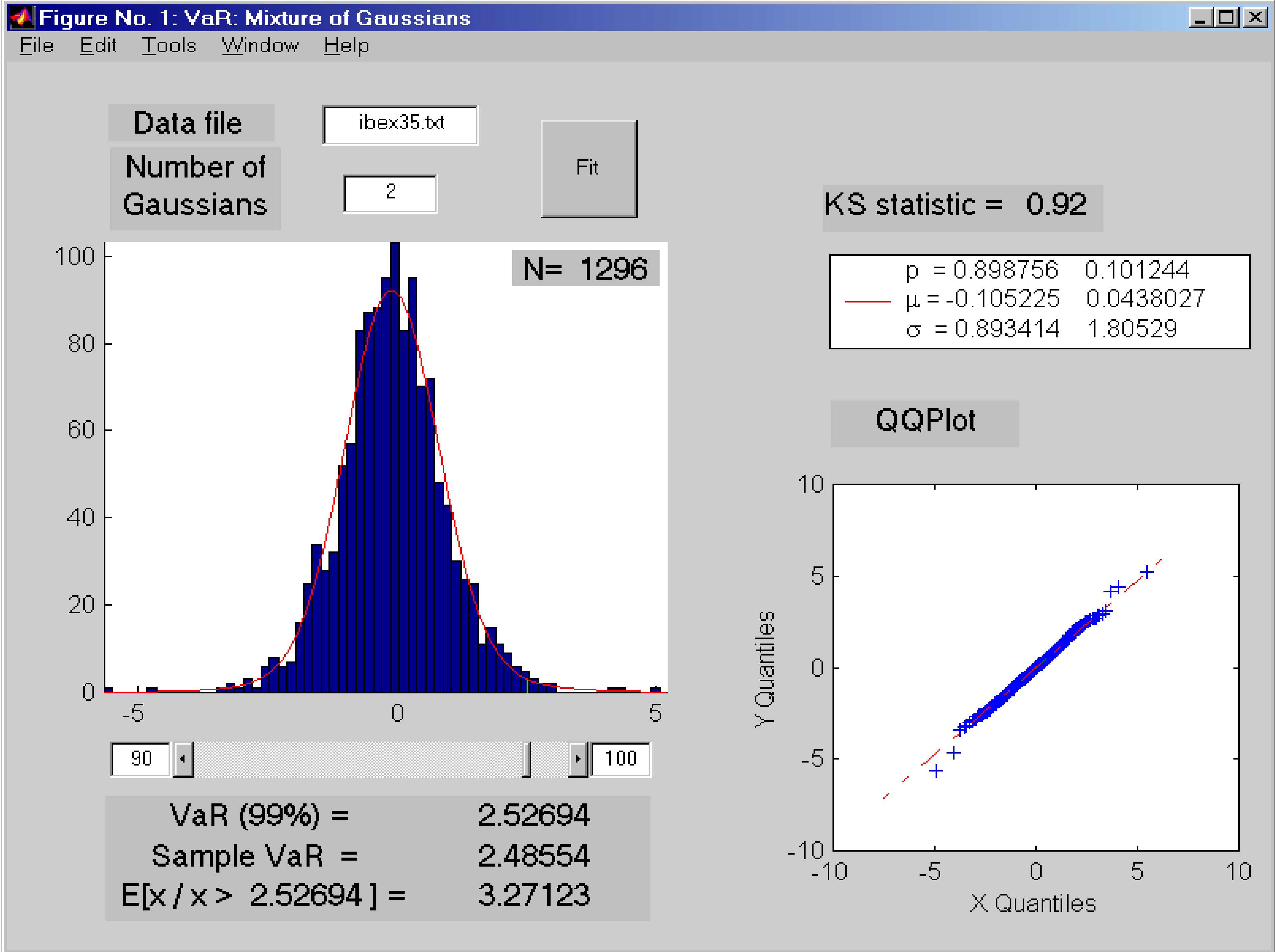


Figure No. 1: VaR: Mixture of Gaussians
File Edit Tools Window Help
Data file
ibex35.txt
Number of Gaussians
2
Fit
KS statistic = 0.92
p = 0.898756 0.101244
μ = -0.105225 0.0438027
σ = 0.893414 1.80529
QQPlot
N= 1296
100
80
60
40
20
0
-5
0
5
90
100
VaR (99%) = 2.52694
Sample VaR = 2.48554
E[x / x > 2.52694] = 3.27123
10
5
0
-5
-10
Y Quantiles
-10
-5
0
5
10
X Quantiles

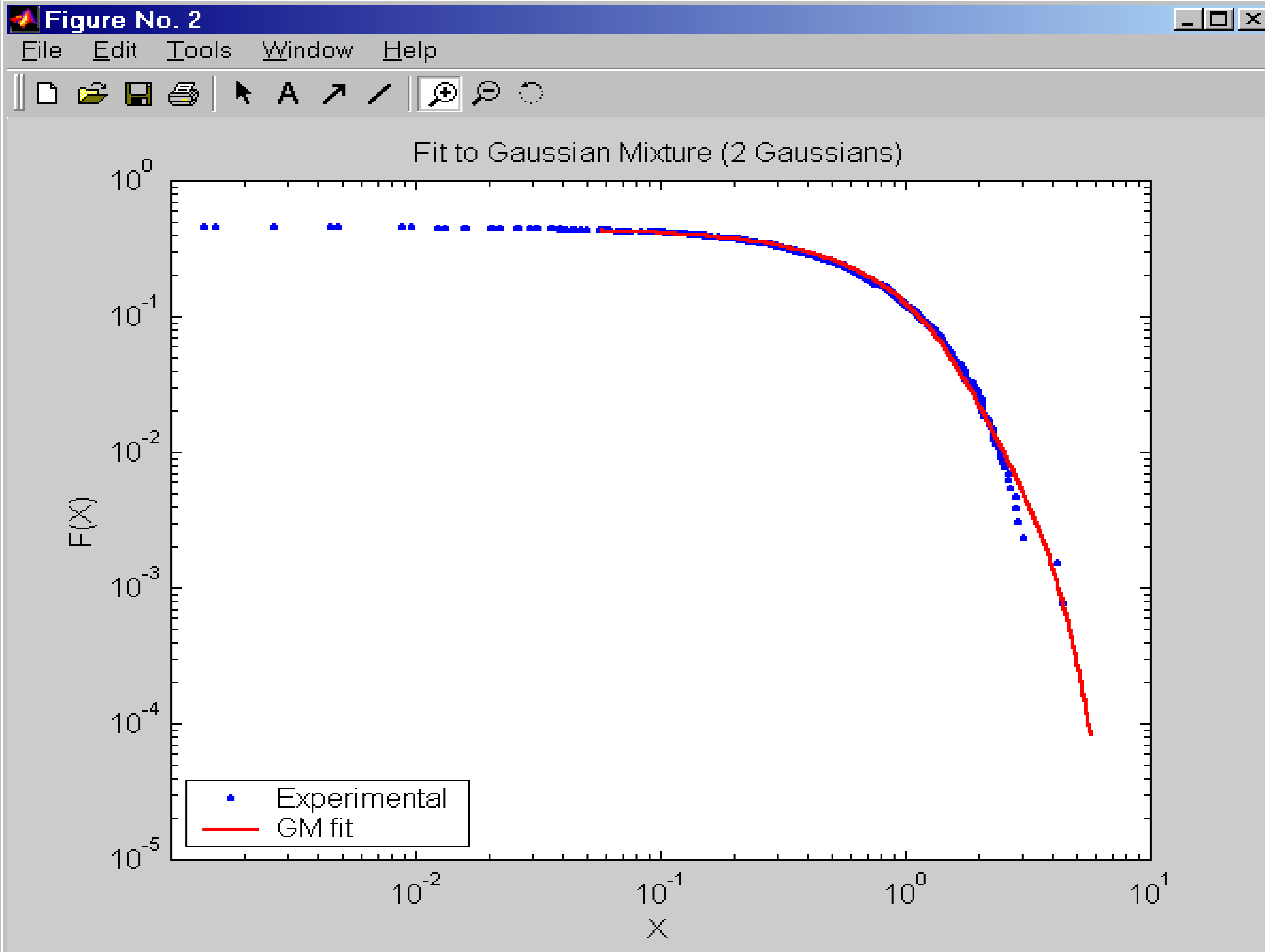


Figure No. 2
File Edit Tools Window Help
Fit to Gaussian Mixture (2 Gaussians)
F(X)
X
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-2}
10^{-1}
10^0
10^1
Experimental
GM fit

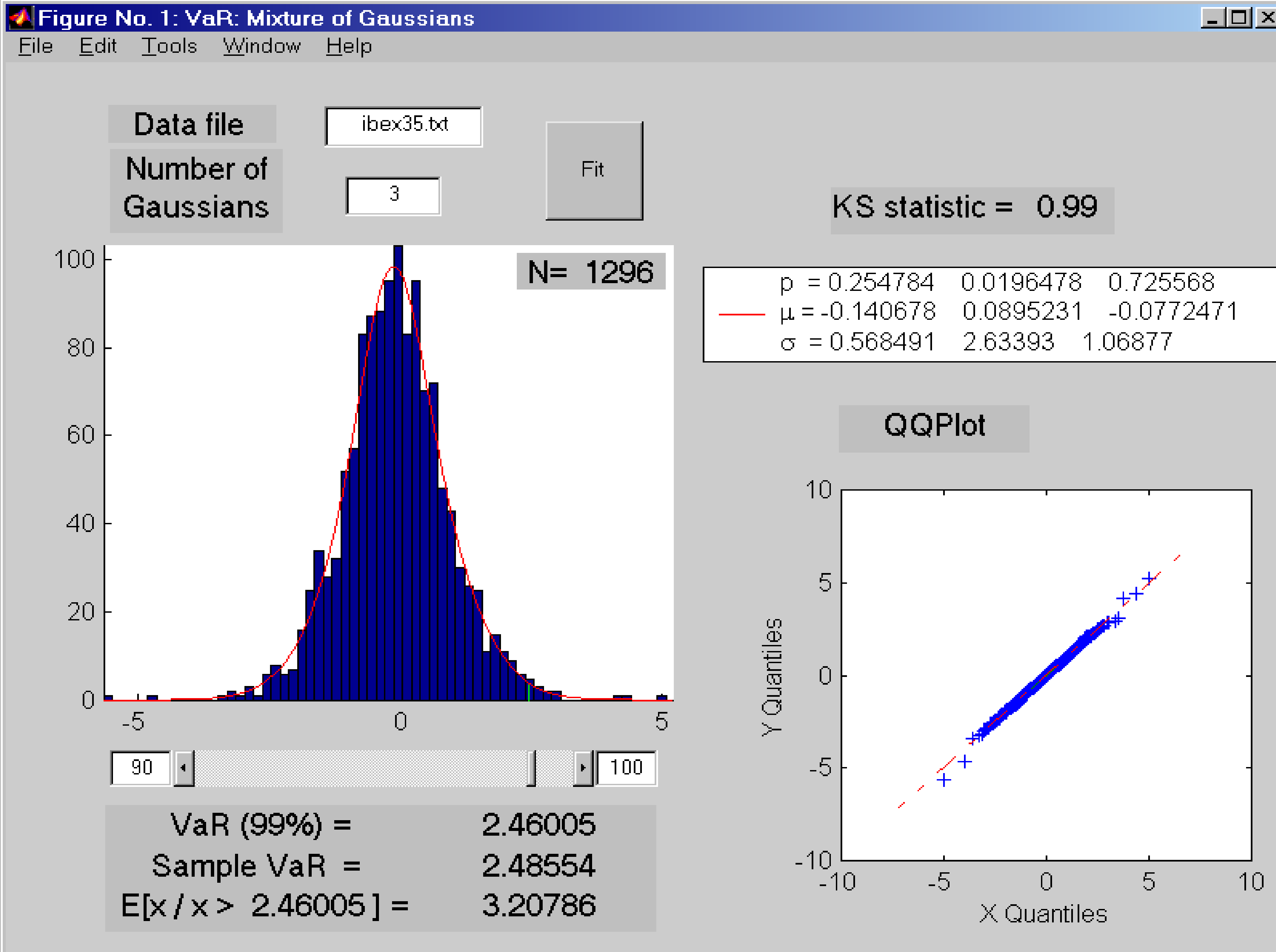


Figure No. 1: VaR: Mixture of Gaussians
File Edit Tools Window Help
Data file
ibex35.txt
Number of Gaussians
3
Fit
KS statistic = 0.99
N= 1296
p = 0.254784 0.0196478 0.725568
μ = -0.140678 0.0895231 -0.0772471
σ = 0.568491 2.63393 1.06877
QQPlot
Y Quantiles
X Quantiles
90
100
VaR (99%) = 2.46005
Sample VaR = 2.48554
E[x / x > 2.46005] = 3.20786

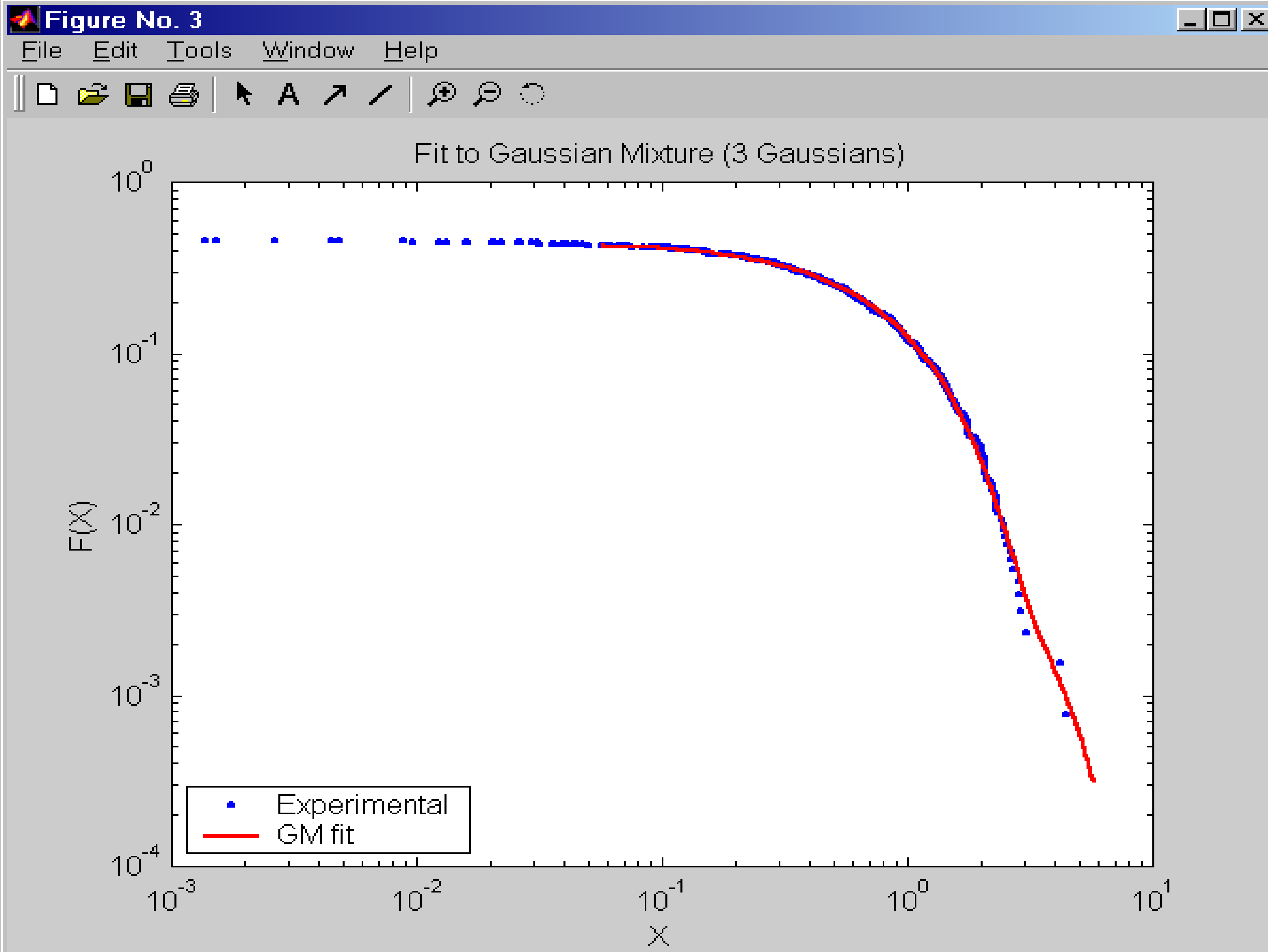


Figure No. 3
File Edit Tools Window Help
Fit to Gaussian Mixture (3 Gaussians)
F(X)
X
Experimental
GM fit
10^0
10^-1
10^-2
10^-3
10^-4
10^-3
10^-2
10^-1
10^0
10^1

Hyperbolic Distribution

- **Probability distribution**

$$P(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1\left(\sqrt{\alpha^2 - \beta^2}\right)} \exp\left(-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)\right)$$

- **Limiting cases**: Normal, Laplace, exponential, generalized inverse Gaussian.
- It is the stationary distribution of a continuous Markov process.

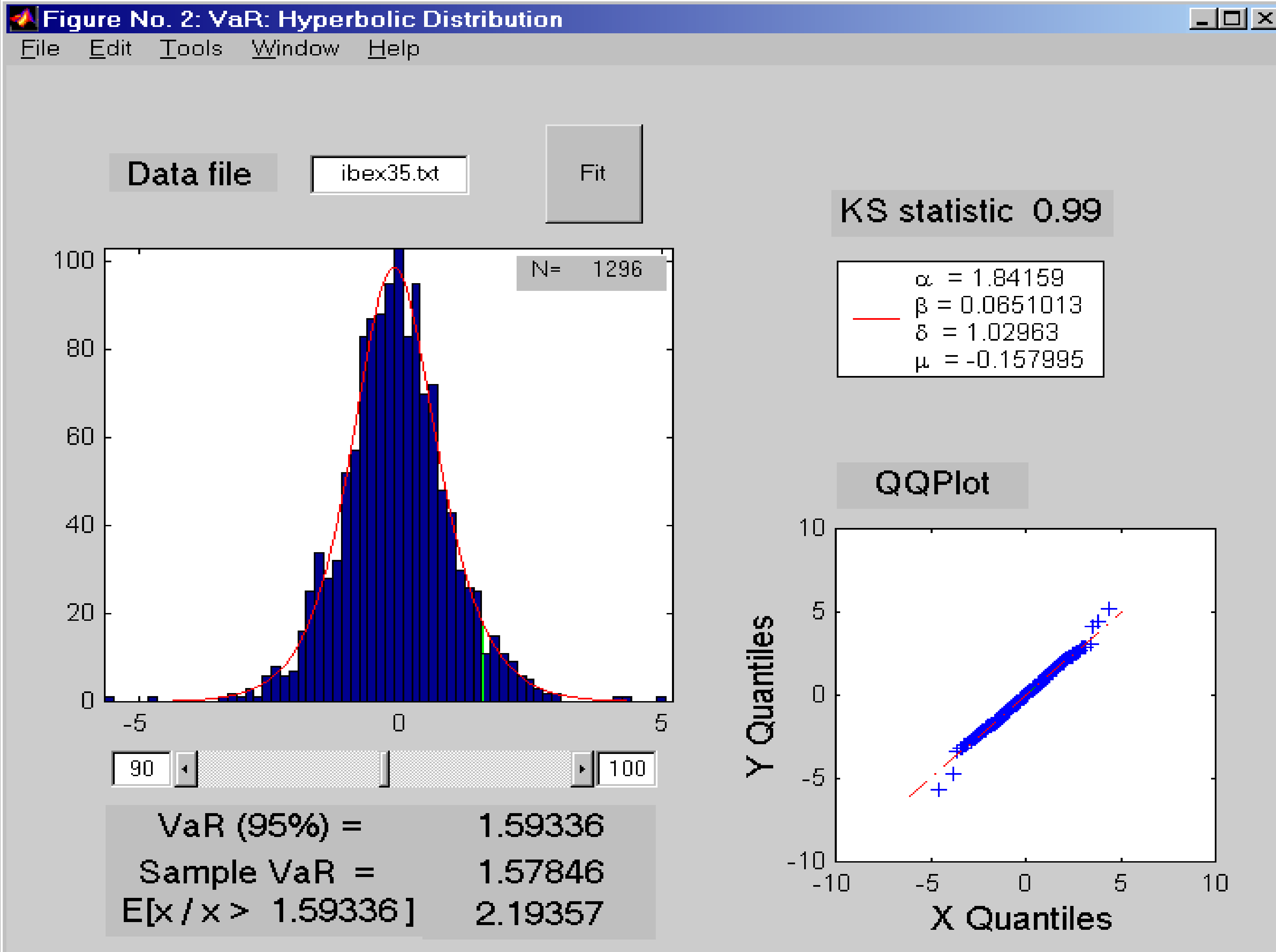


Figure No. 2: VaR: Hyperbolic Distribution
File Edit Tools Window Help
Data file
ibex35.txt
Fit
KS statistic 0.99
α = 1.84159
β = 0.0651013
δ = 1.02963
μ = -0.157995
N= 1296
100
80
60
40
20
0
-5
0
5
90
100
VaR (95%) = 1.59336
Sample VaR = 1.57846
E[x / x > 1.59336] 2.19357
QQPlot
10
5
0
-5
-10
Y Quantiles
X Quantiles

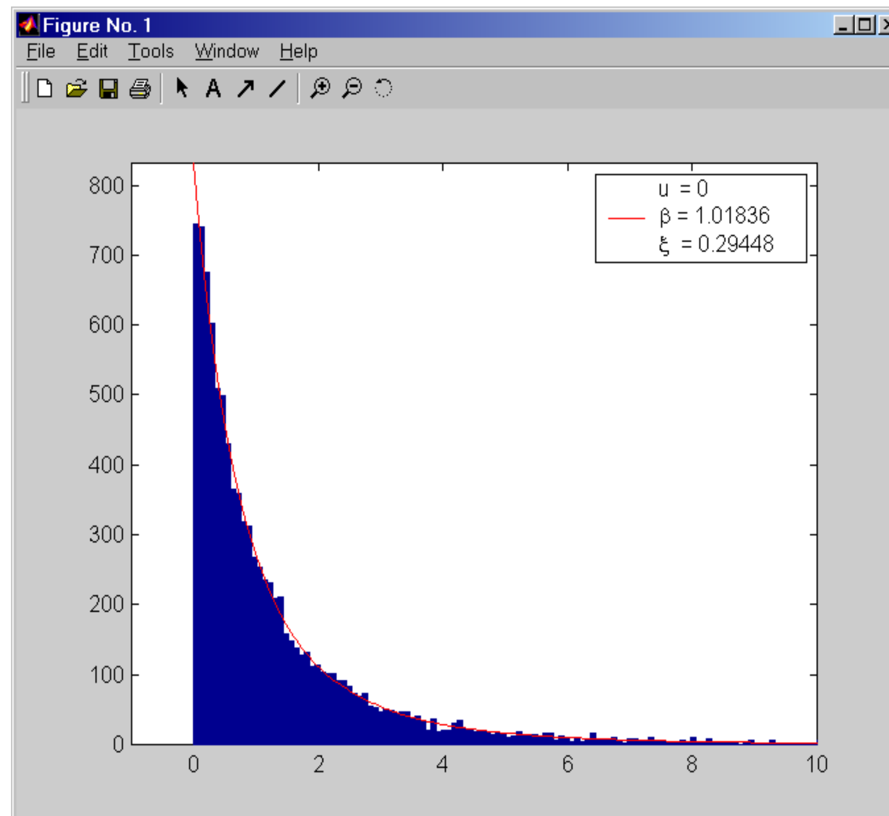
VaR: Advantages / drawbacks

- **Advantages**:
  - Summarizes exposure to market risk in a single number.
  - Simple, intuitive, easy to understand.
- **Drawbacks**:
  - It is not subadditive for heavy-tailed distributions.
  - Problems to capture events in the tails.
  - Problems with time aggregation.

Beyond VaR: Shortfall

Generalized Pareto Distribution

$$P(x) = \frac{1}{\beta}\left(1+\frac{\xi}{\beta}(x-u)_+\right)^{-\left(1+\frac{1}{\xi}\right)}$$



<u>Objective</u>: Fit the **tails** of the distribution, assuming **algebraic decay**.

Estimating Shortfall

- **Definition:** $$Shortfall \equiv E\left[x / x > u\right]$$

- **Mean excess** $$e(u) = \frac{1}{N_u} \sum_{n=1}^{N_u} (x_n - u)_+$$

  - Rank-ordered points: $x_1 \geq x_2 \geq \ldots \geq x_N$
  - Choose u so that mean excess is <u>lineal</u> in u.
  - <u>Parametric fit</u> to GPD by log-likelihood maximization.
  - <u>Find shortfall</u> from GPD.

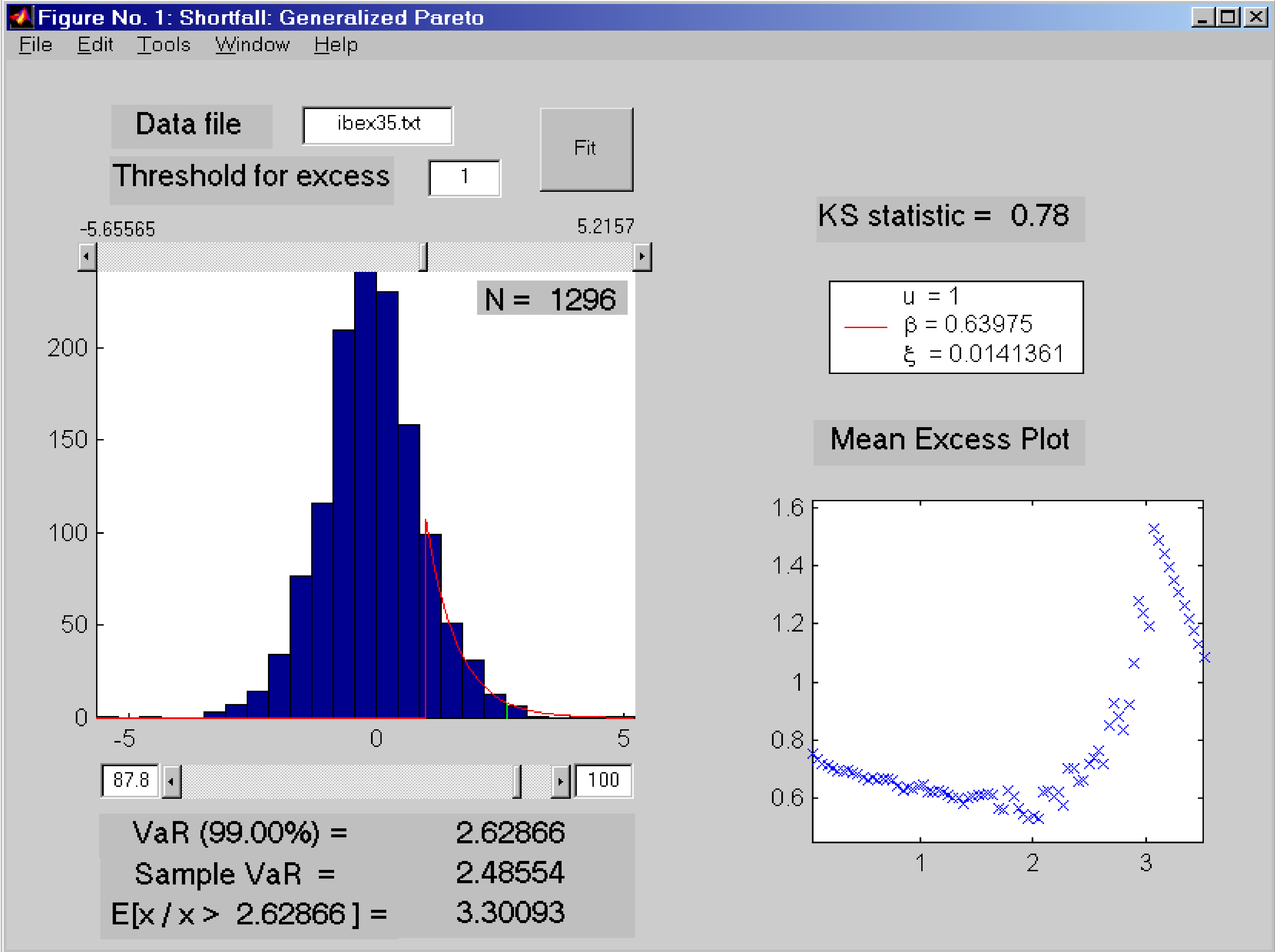


Figure No. 1: Shortfall: Generalized Pareto
File Edit Tools Window Help
Data file
ibex35.txt
Fit
Threshold for excess
1
-5.65565
5.2157
N = 1296
87.8
100
VaR (99.00%) = 2.62866
Sample VaR = 2.48554
E[x / x > 2.62866] = 3.30093
KS statistic = 0.78
u = 1
β = 0.63975
ξ = 0.0141361
Mean Excess Plot

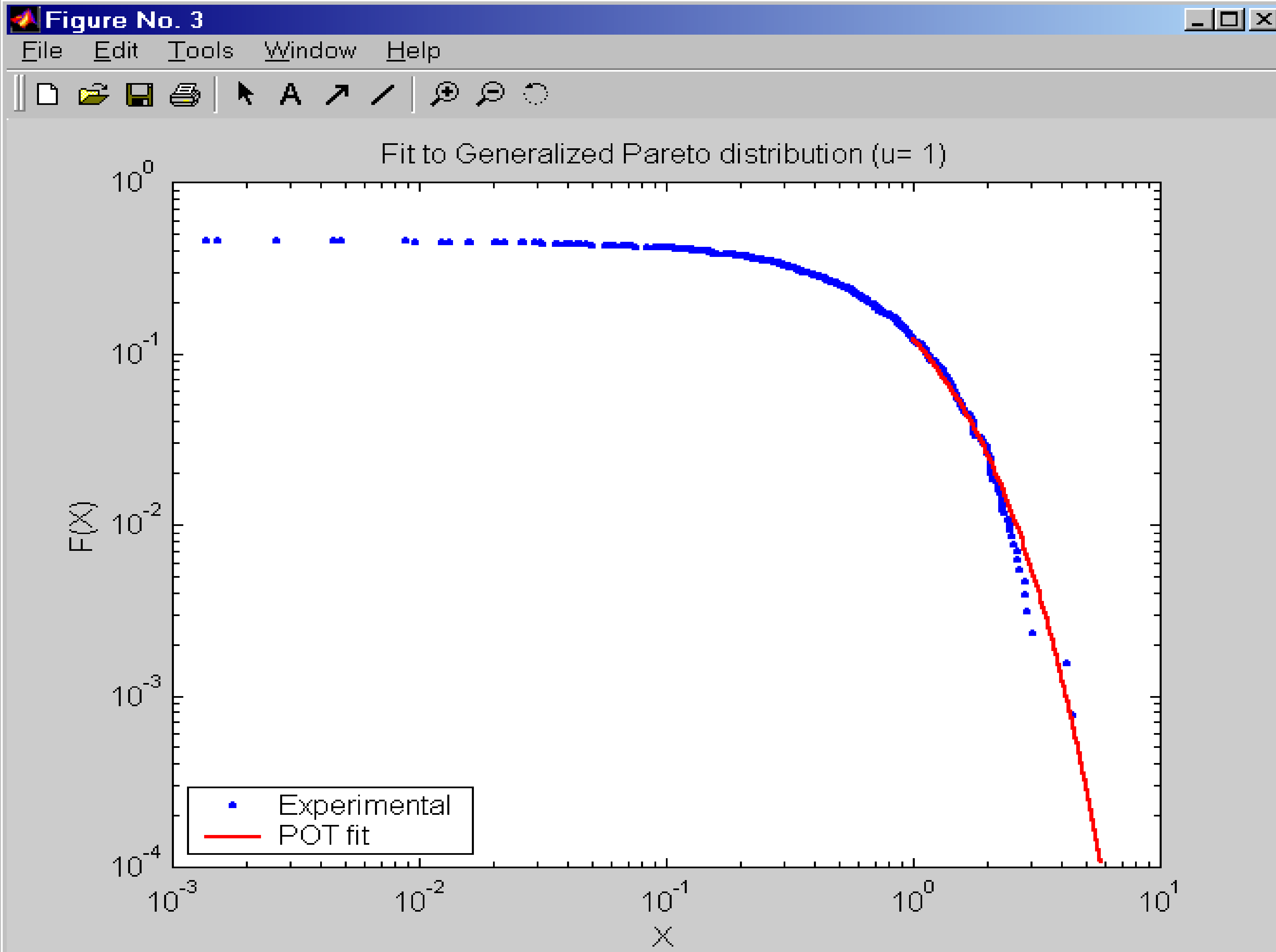


Figure No. 3
File
Edit
Tools
Window
Help
Fit to Generalized Pareto distribution (u= 1)
F(X)
X
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
Experimental
POT fit

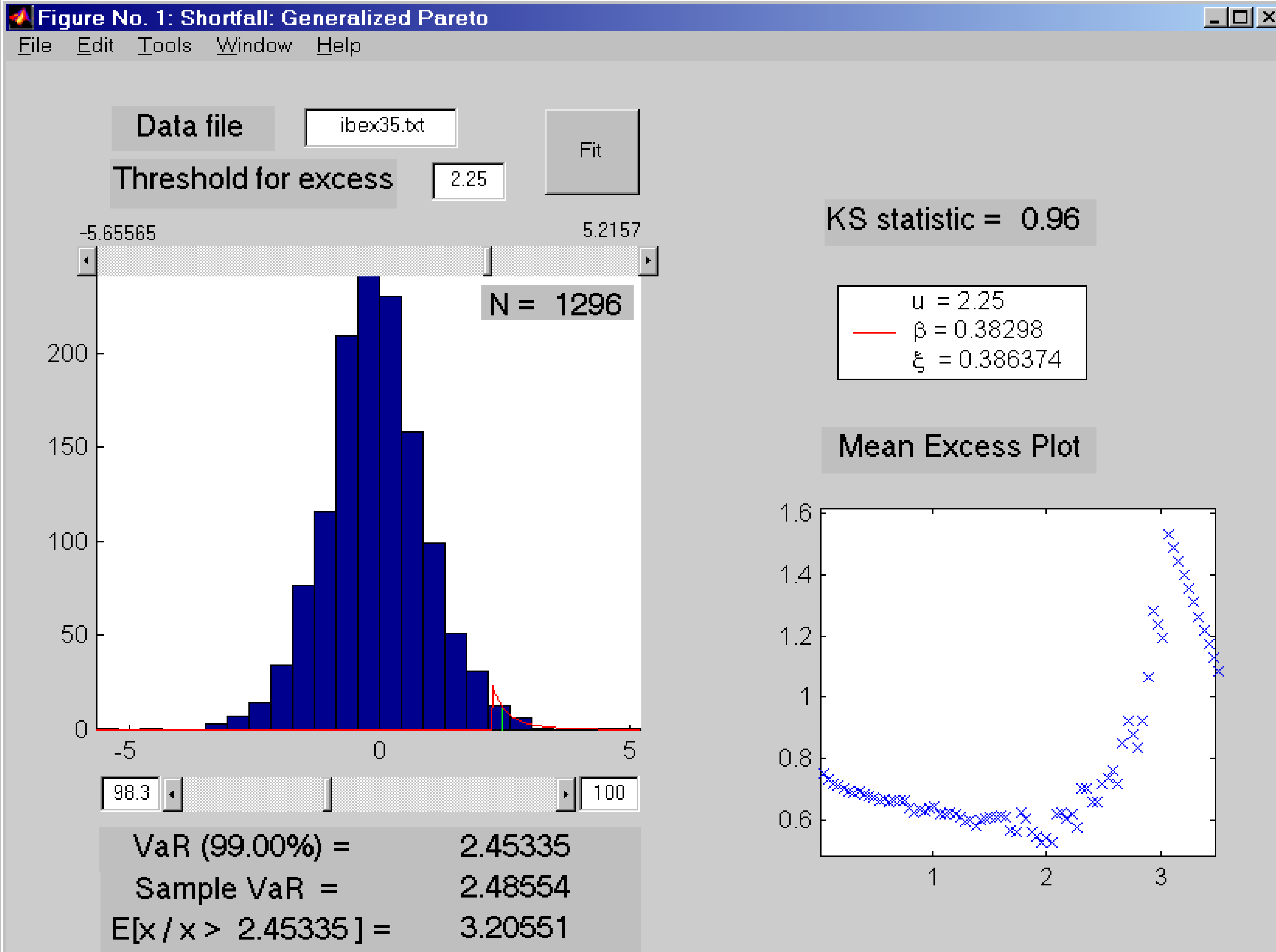


Figure No. 1: Shortfall: Generalized Pareto
File Edit Tools Window Help
Data file
ibex35.txt
Threshold for excess
2.25
Fit
-5.65565
5.2157
N = 1296
200
150
100
50
0
-5
0
5
98.3
100
VaR (99.00%) = 2.45335
Sample VaR = 2.48554
E[x / x > 2.45335] = 3.20551
KS statistic = 0.96
u = 2.25
β = 0.38298
ξ = 0.386374
Mean Excess Plot
1.6
1.4
1.2
1
0.8
0.6
1
2
3

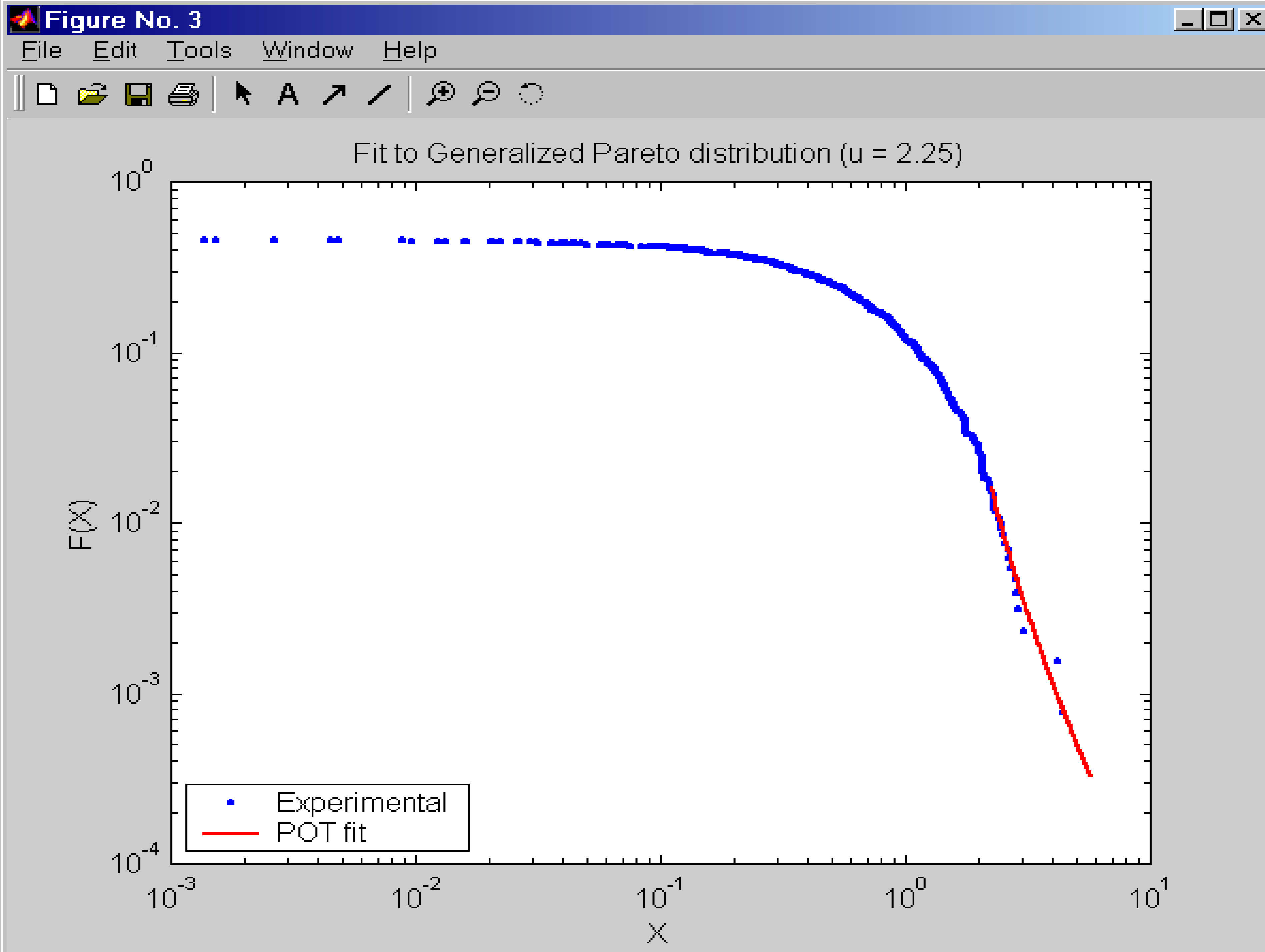


Figure No. 3
File
Edit
Tools
Window
Help
Fit to Generalized Pareto distribution (u = 2.25)
F(X)
X
10^0
10^-1
10^-2
10^-3
10^-4
10^-3
10^-2
10^-1
10^0
10^1
Experimental
POT fit

Shortfall in finance

- The value of ξ in finance is typically in the interval [0.2,0.7]
- **Tail index**: $\alpha = 1/\xi$; $1.5 < \alpha < 5$

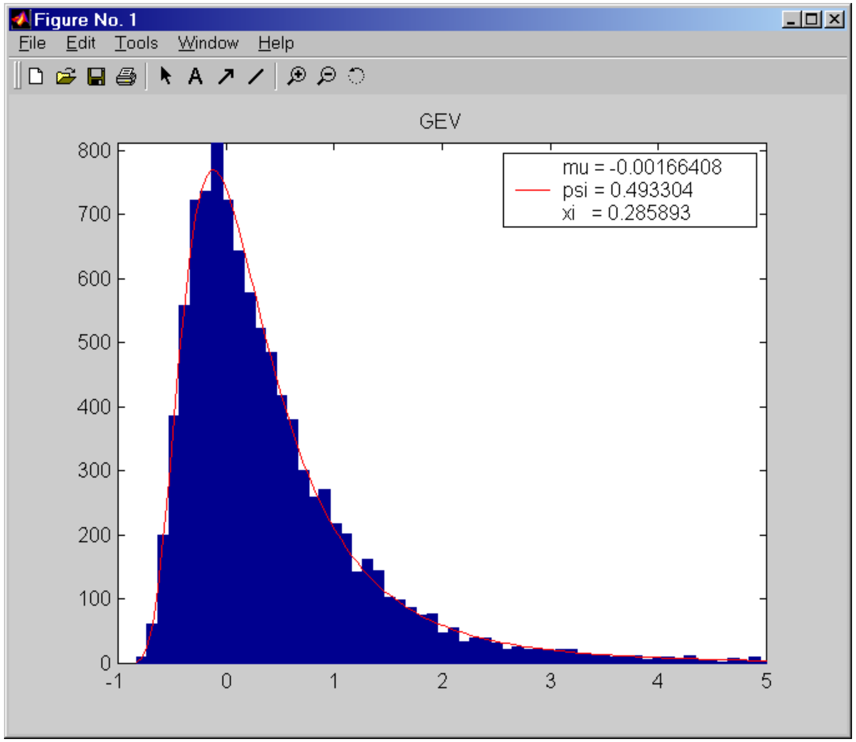
- **VaR / ShortFall**

$$E[x / x > VaR] \approx \frac{\alpha}{\alpha - 1} VaR$$

Beyond VaR: MaxVar

Generalized Extreme Value Distribution

$$P(x)=\frac{1}{\psi}\left(1+\frac{\xi}{\psi}(x-\mu)\right)^{-\left(1+\frac{1}{\xi}\right)} e^{-\left(1+\frac{\xi}{\psi}(x-\mu)\right)^{-\frac{1}{\xi}}}$$

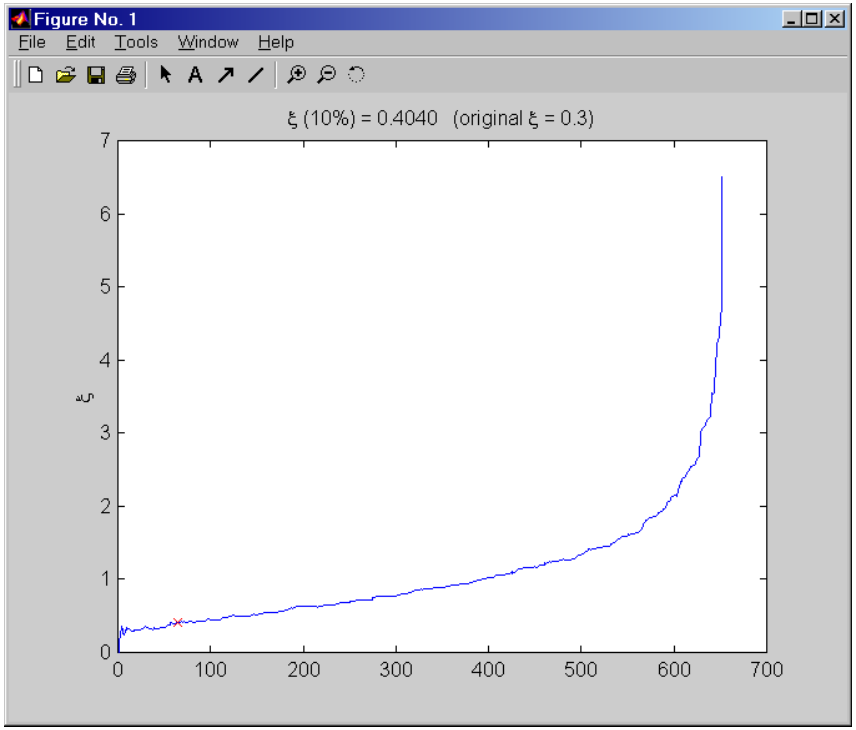


Objective: Fit **maxima** of the distribution, assuming **independent events**.

Estimate of shape parameter

- **Hill Estimator** for ξ
  - Rank-ordered points: $x_1 \geq x_2 \geq \dots \geq x_N$
  - $k >> 1; \ k << N$

$$\xi_{Hill} \equiv \frac{1}{k}\sum_{j=1}^{k}\log\frac{x_j}{x_k}$$



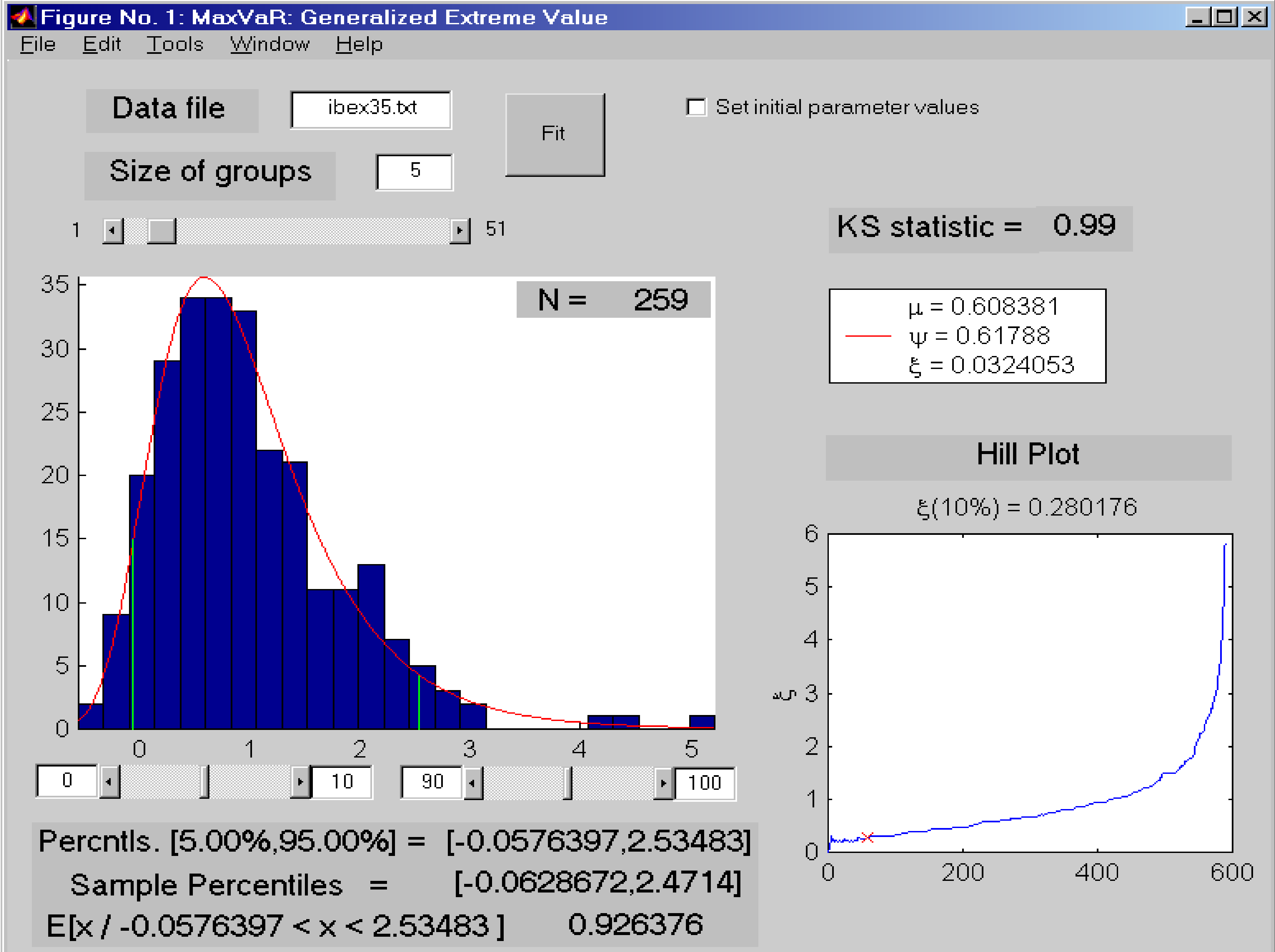


Figure No. 1: MaxVaR: Generalized Extreme Value
File Edit Tools Window Help
Data file
ibex35.txt
Fit
Set initial parameter values
Size of groups
5
1
51
KS statistic = 0.99
N = 259
μ = 0.608381
ψ = 0.61788
ξ = 0.0324053
Hill Plot
ξ(10%) = 0.280176
ξ
0
10
90
100
Percntls. [5.00%,95.00%] = [-0.0576397,2.53483]
Sample Percentiles = [-0.0628672,2.4714]
E[x / -0.0576397 < x < 2.53483] 0.926376

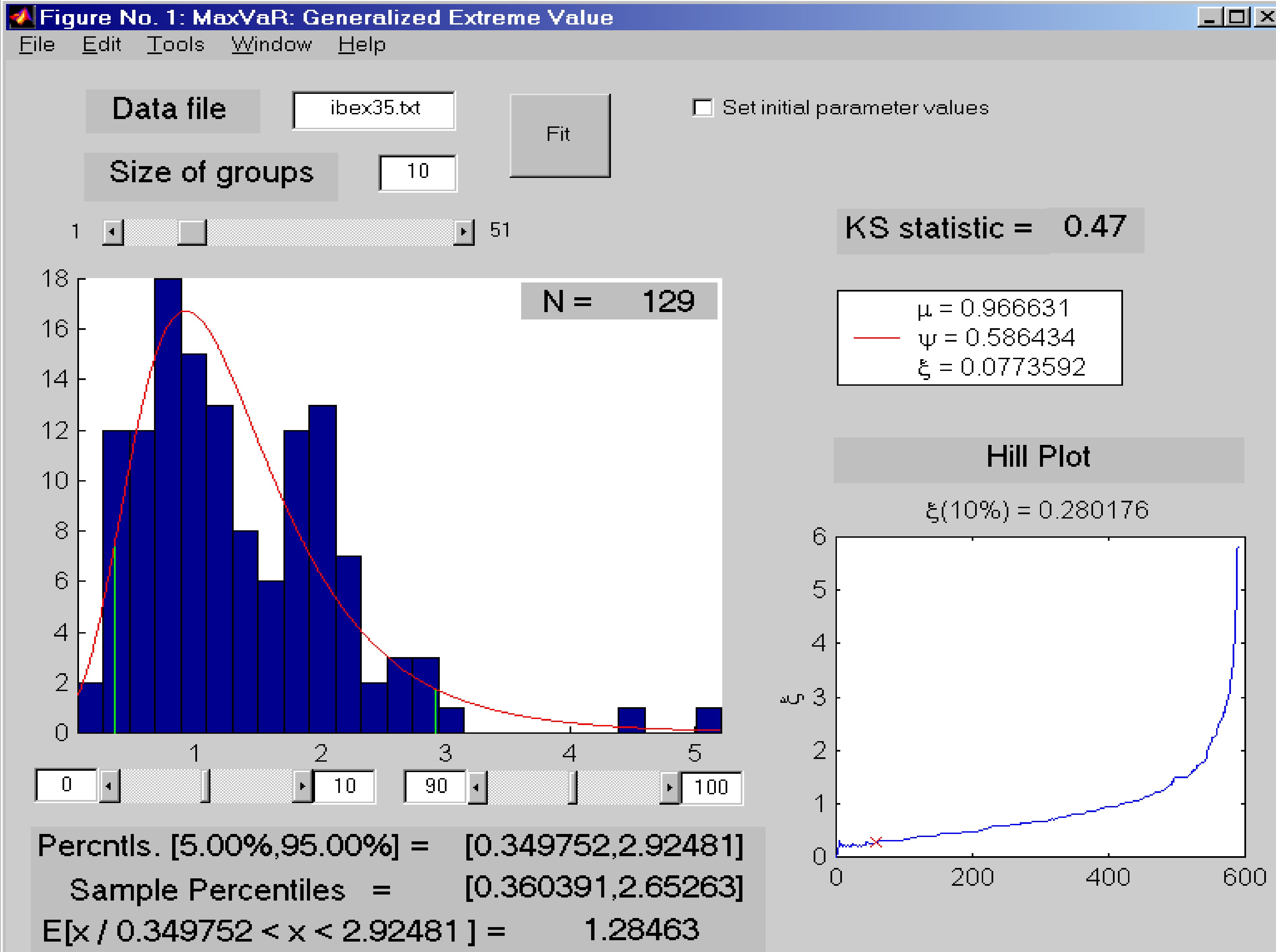


Figure No. 1: MaxVaR: Generalized Extreme Value
File Edit Tools Window Help
Data file
ibex35.txt
Fit
Set initial parameter values
Size of groups
10
1
51
KS statistic = 0.47
N = 129
μ = 0.966631
ψ = 0.586434
ξ = 0.0773592
Hill Plot
ξ(10%) = 0.280176
ξ
0
10
90
100
Percntls. [5.00%,95.00%] = [0.349752,2.92481]
Sample Percentiles = [0.360391,2.65263]
E[x / 0.349752 < x < 2.92481] = 1.28463

MaxVar: A balance

- Based in **universal properties** of distributions of maxima of iid random variables.
- Accounts for **extreme events** (beyond sample).
- Does **not assume** an ***a priori*** functional **form** for the original probability distribution.
- Despite the fact that the parameters of the GEV distribution vary greatly, the **percentiles** measured **are** fairly **stable**.

Summary IBEX35 (I)

- Comparison between different measures of VaR

	KS Statistic	95%	99%	99.9%	99.99%
Sample VaR		1.58	2.49	4.40	5.22
GM1	0.12	1.60	2.29	3.36	3.72
GM2	0.92	1.54	2.53	4.25	5.63
GM3	0.99	1.58	2.46	4.41	6.86
HYP	0.99	1.59	2.57	3.91	5.30
GP (u=1)	0.78	1.57	2.63	4.18	5.78
GP (u=2.25)	0.96	***	2.45	4.16	8.34

Summary IBEX35 (II)

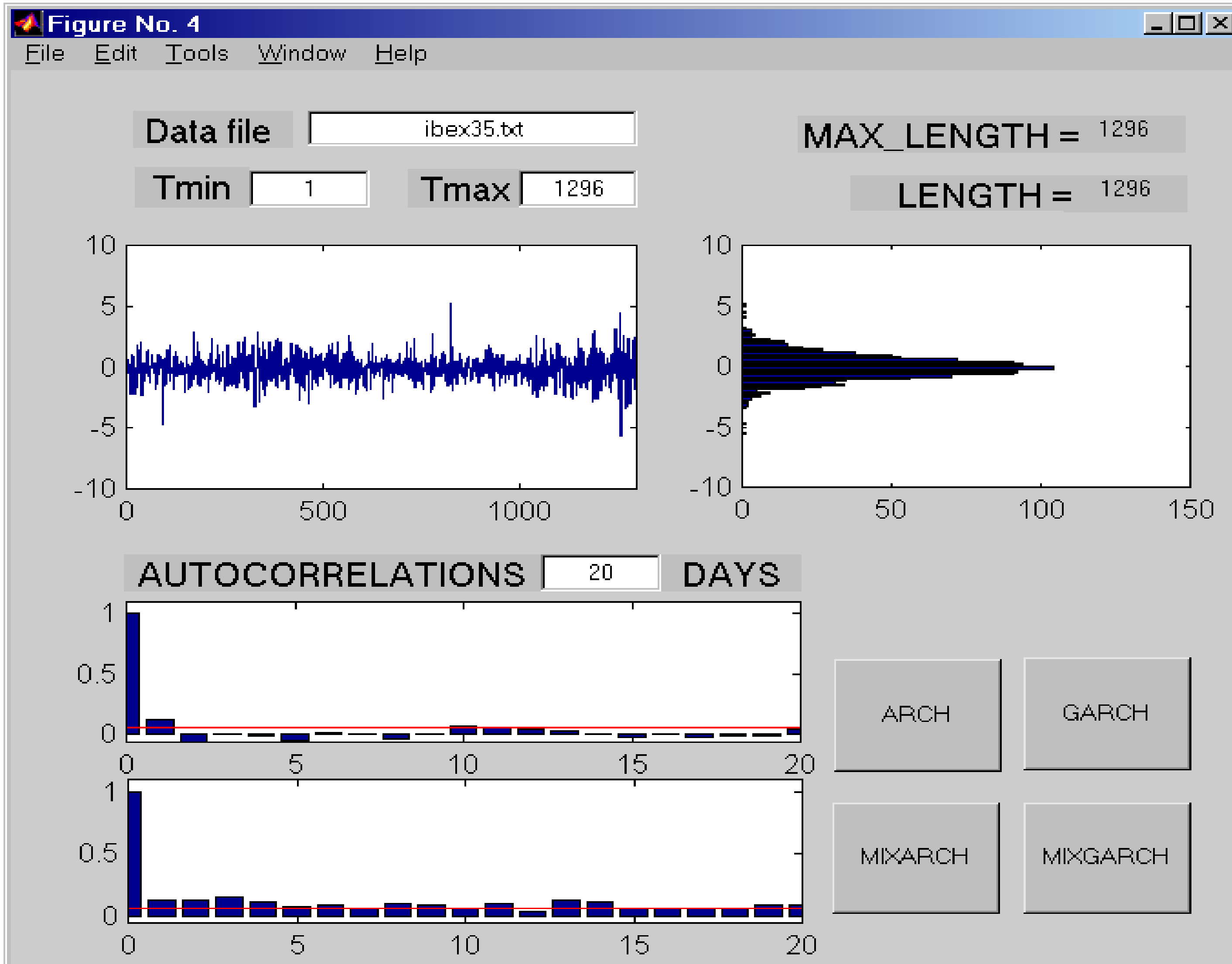
- Comparison between different measures of **Shortfall**

	KS Statistic	95%	99%	99.9%	99.99%
GM1	0.12	2.02	2.64	3.07	3.97
GM2	0.92	2.16	3.27	4.86	6.13
GM3	0.99	2.17	3.21	5.51	7.69
HYP	0.99	2.20	3.15	4.49	5.80
GP (u=1)	0.78	2.23	3.30	4.87	6.50
GP (u=2.25)	0.96	***	3.20	5.98	12.79

Real markets

- In practice markets are
  - Not efficient:
    - Memory effects (short/long term?).
  - Very unpredictable (at least sometimes)
    - **Extreme events** are more frequent than what the normal models predict.
    - Occurrence of **crashes**.
    - **Abrupt changes** in economic paradigm.

Memory effects (IBEX 35)



Stationary random time series

- Consider the time series

$$X_0, X_1, X_2, \ldots X_{t-1}, X_t, \ldots$$

  - The series exhibits randomness.
  - The process is covariance-stationary:
    - **Mean** is time independent

$$E[X_t] = \mu$$

    - **Autocovariance** is independent of time-translations

$$E[(X_{t+\tau} - \mu)(X_t - \mu)] = \gamma_\tau$$

Conditional heteroskedasticity

- Model the time-structure of the volatility with an ARCH process

$$X_t = \phi^{+} \cdot \mathbf{X}_t^{(m)} + u_t$$

$$u_t = \sigma_t Z_t$$

- The quantities Z_t are assumed to be iidrv's.
- The volatility follows an AR(q) process

$$\sigma_t^2 = \kappa + \alpha^{+} \cdot \left[\mathbf{u}^2\right]_t^{[q]}$$

AR(1) / ARCH(1) for IBEX35

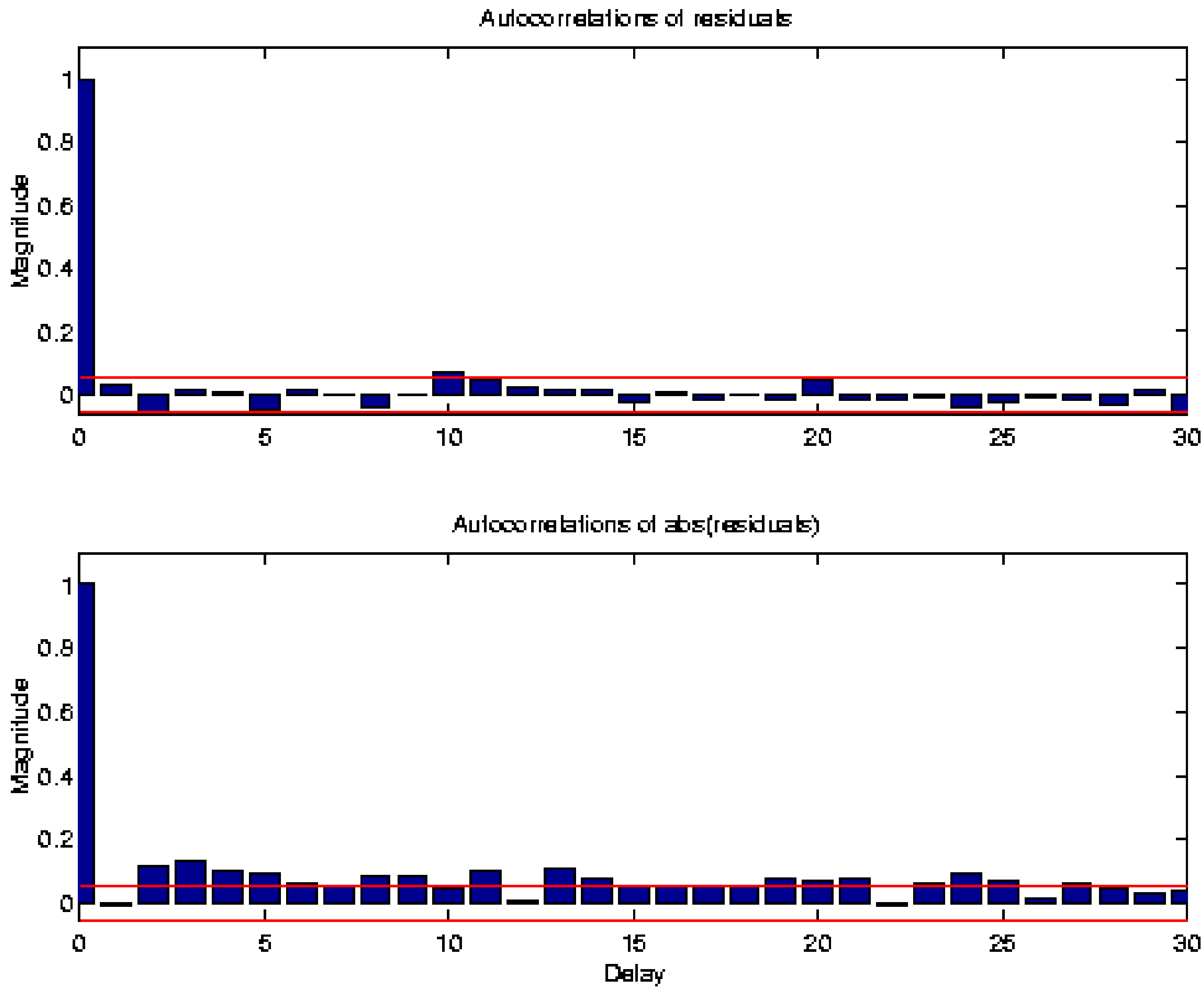
- The maximum-likelihood fit of the time-series IBEX35 yields the model

$$\hat{X}_t = 0.1129\,\hat{X}_{t-1} + \sigma_t Z_t$$

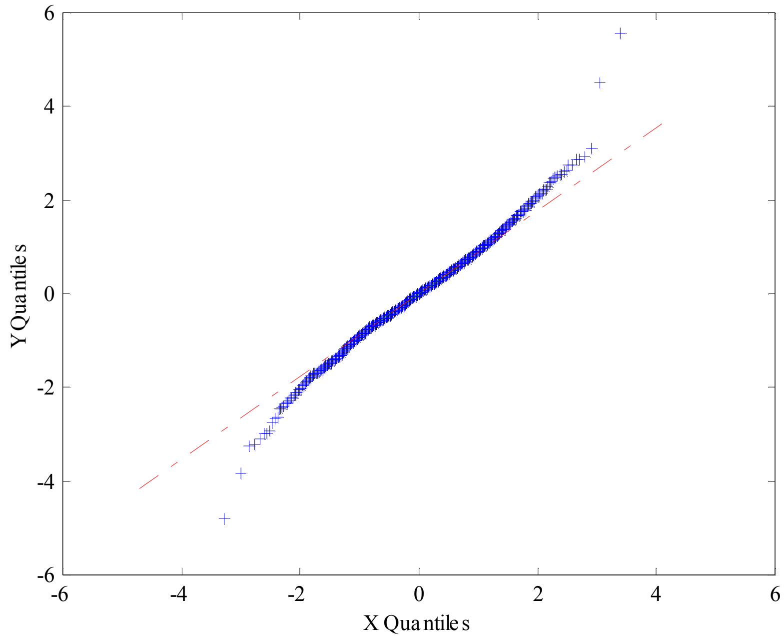
$$\sigma_t^2 = 0.9097 + 0.1118\left(\hat{X}_{t-1} - 0.1129\,\hat{X}_{t-2}\right)^2$$

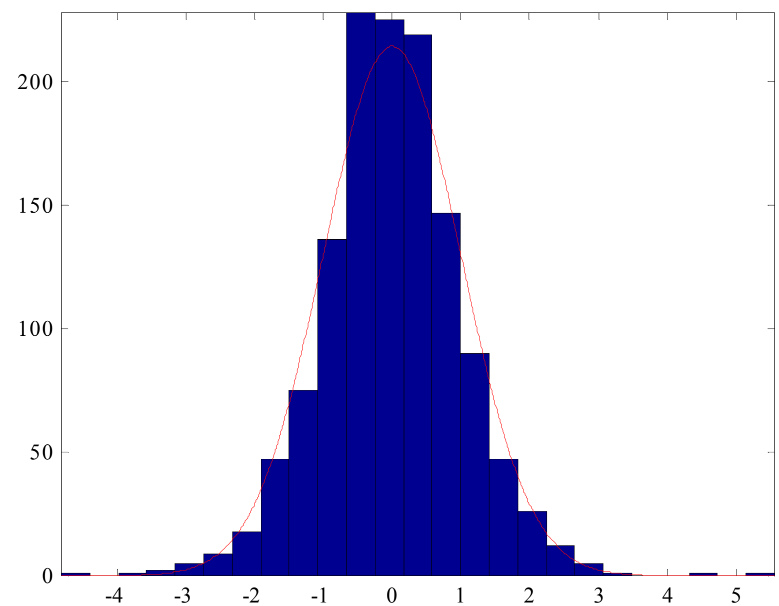
➤ The quantities Z_t are assumed to follow a N(0,1) distribution.

Residual correlations: ARCH(1)



Normality hypothesis: ARCH(1)





KS Test = 0.12

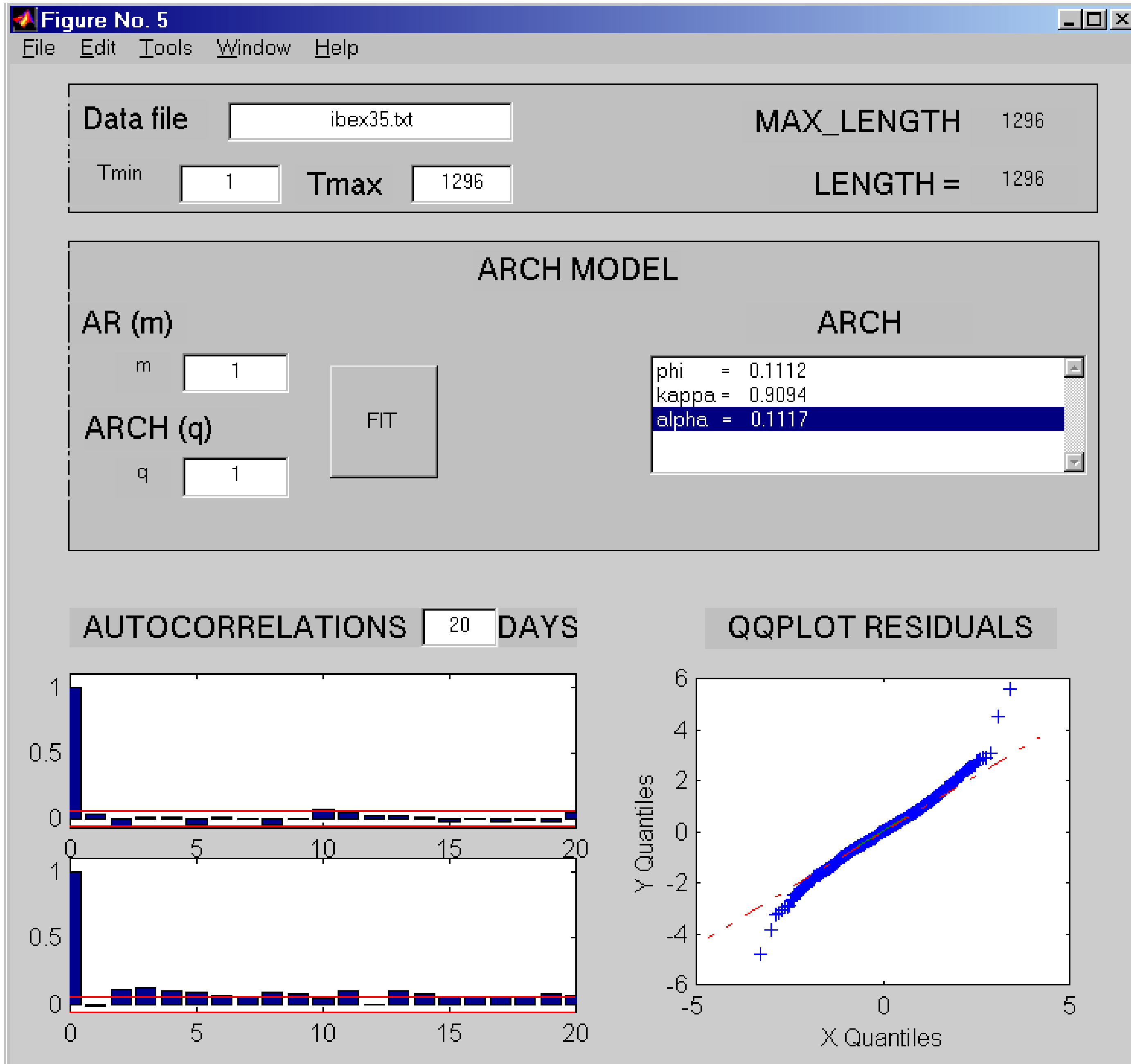


Figure No. 5
File Edit Tools Window Help
Data file
ibex35.txt
MAX_LENGTH
1296
Tmin
1
Tmax
1296
LENGTH =
1296
ARCH MODEL
AR (m)
m
1
ARCH (q)
q
1
FIT
ARCH
phi = 0.1112
kappa = 0.9094
alpha = 0.1117
AUTOCORRELATIONS
20
DAYS
QQPLOT RESIDUALS
Y Quantiles
X Quantiles

AR(1) / GARCH(1,1) for IBEX35

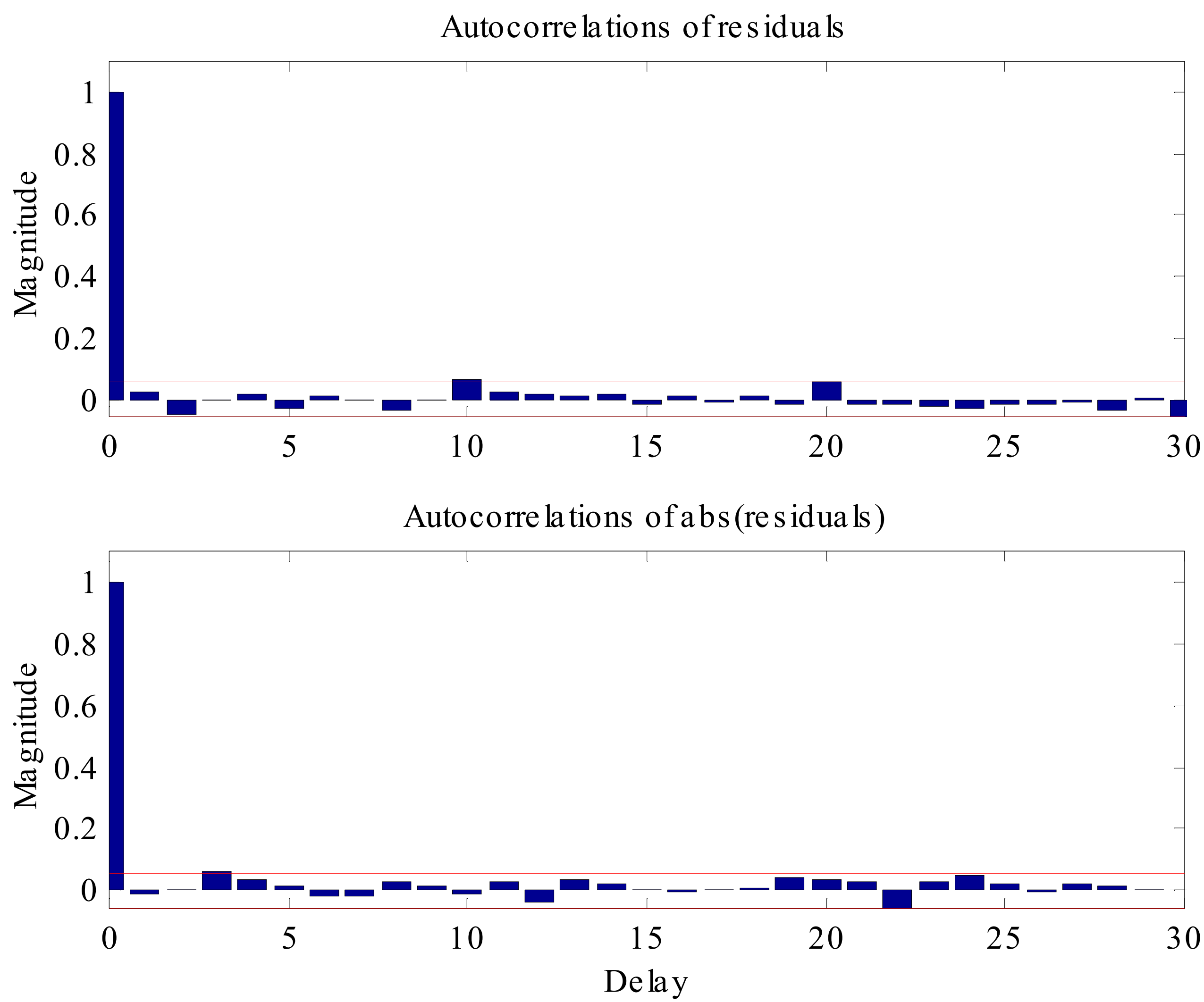
- The maximum-likelihood fit of the time-series IBEX35 yields the model

$$\hat{X}_t = 0.1358\hat{X}_{t-1} + \sigma_t Z_t$$

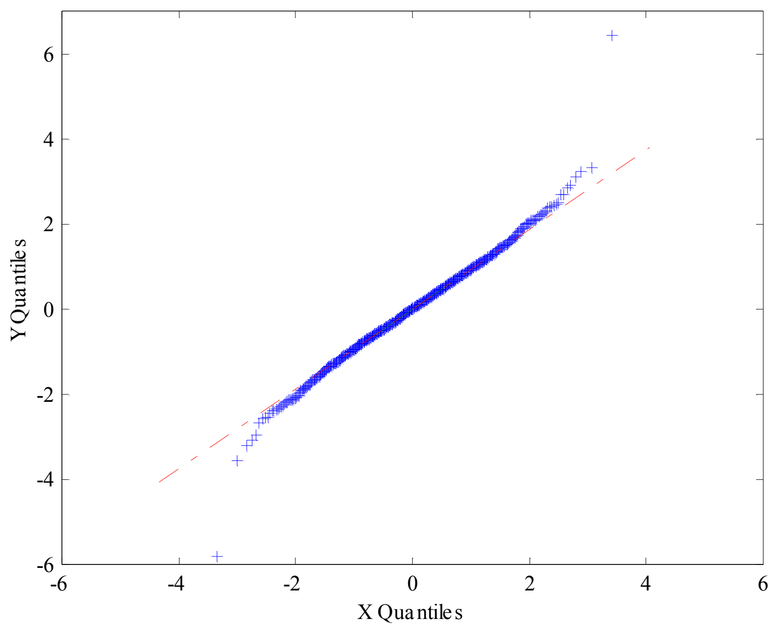
$$\sigma_t^2 = 0.0527 + 0.0755\left(\hat{X}_{t-1} - 0.1358\hat{X}_{t-2}\right)^2 + 0.8733\sigma_{t-1}^2$$

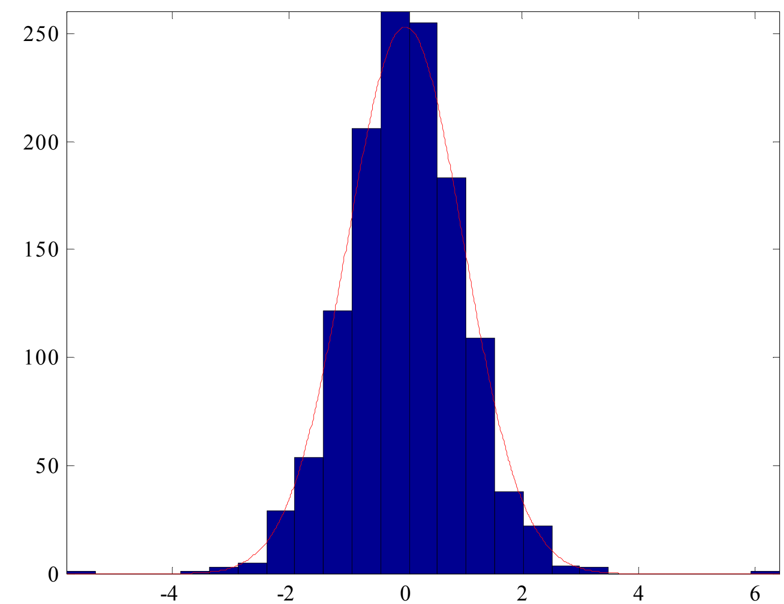
  - The quantities Z_t are assumed to follow a N(0,1) distribution.

Residual correlations: GARCH



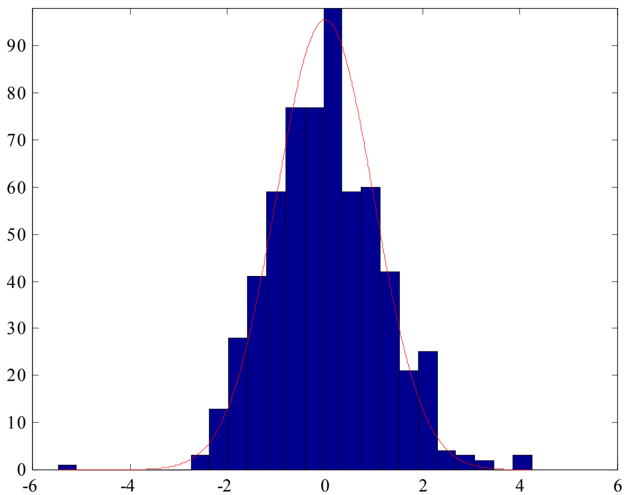
Normality hypothesis: GARCH(1,1)

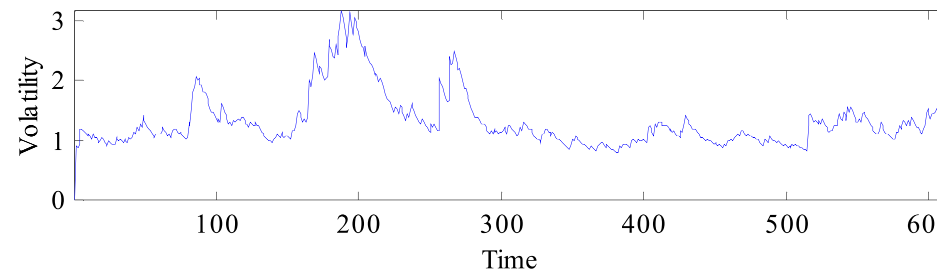


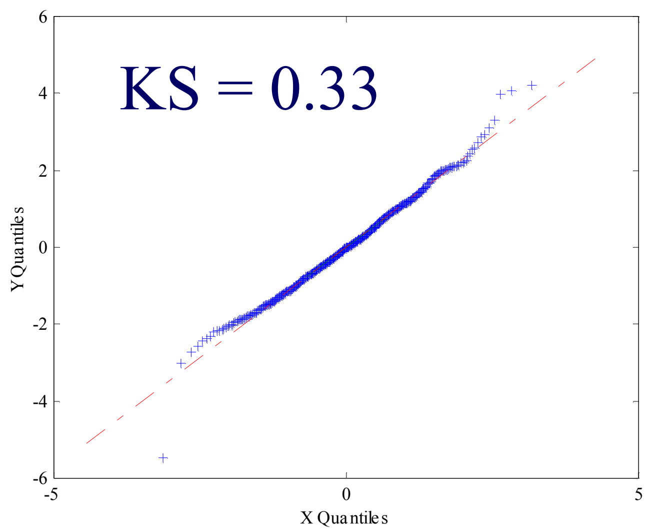


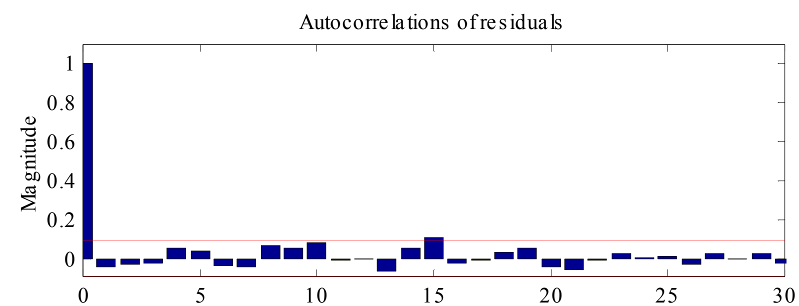
KS Test = 0.56

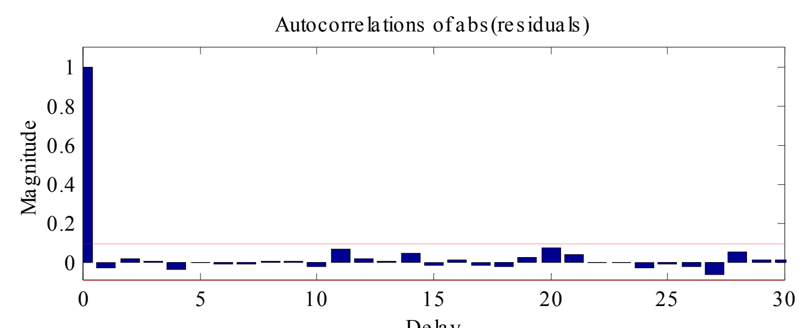
Test Data




Volatility
Time

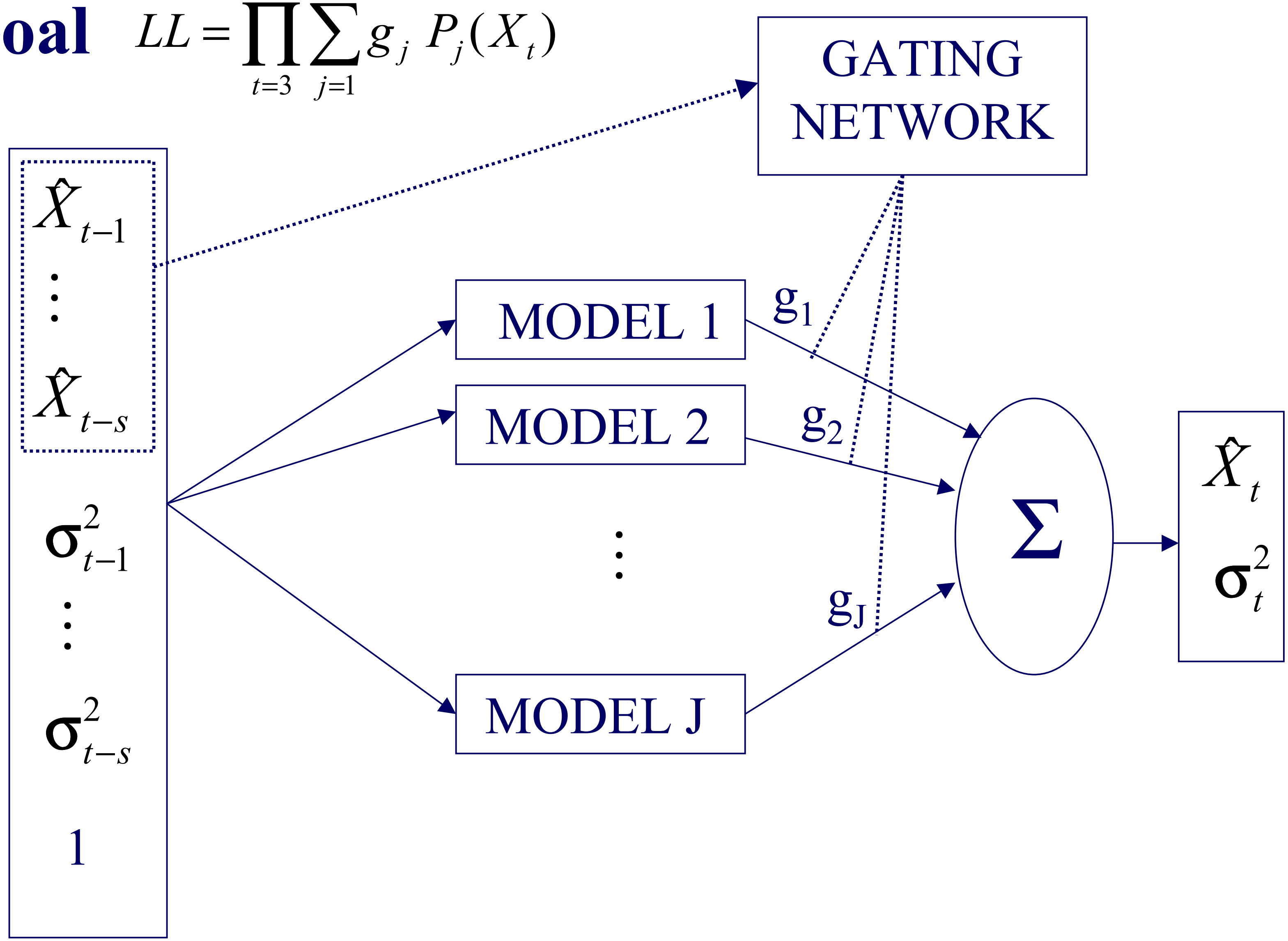

KS = 0.33
Y Quantiles
X Quantiles


Autocorrelations of residuals
Magnitude


Autocorrelations of abs(residuals)
Magnitude
Delay

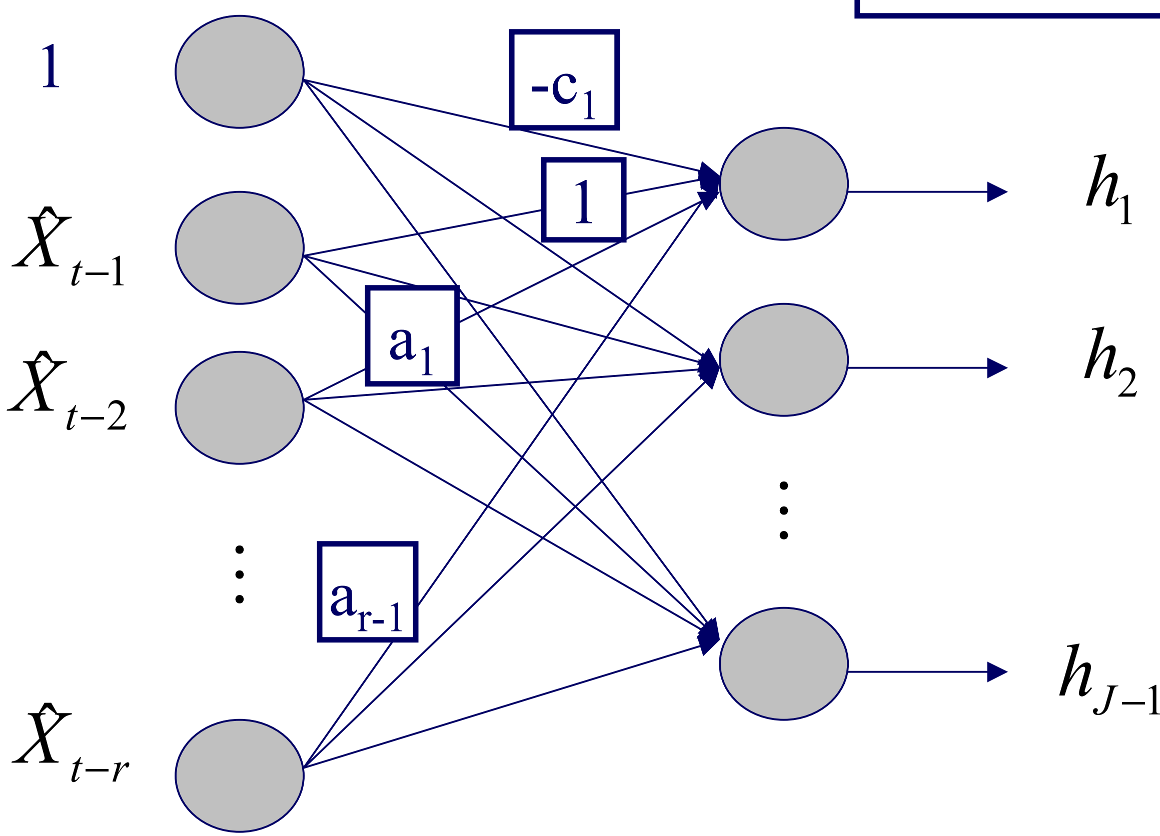
Mixture model

Goal $LL = \prod_{t=3}^{T} \sum_{j=1}^{J} g_j \, P_j(X_t)$



Gating Network

$$h_i = \exp\left[b_i \left(\hat{X}_{t-1} + \sum_{k=1}^{r-1} a_i \hat{X}_{t-k-1} - c_i \right) \right]$$



Probabilities

$$g_i = \frac{h_i}{1 + \sum_{j=1}^{J-1} h_j}; \quad i = 1,2,\ldots(J-1) \quad g_J = 1 - \sum_{j=1}^{J-1} g_j$$

Mixture of ARCH processes

- MixARCH $$X_t = \phi_{[i]}^{+} \cdot \mathbf{X}_t^{[m]} + u_{[i]}(t),$$

$$\textit{with probability} \;\; g_{[i]}(\mathbf{X}_t^{[r]}, \theta_{[i]})$$

  - ➤The model for the residuals is

$$u_{[i]}(t) = \sigma_{[i]}(t) Z_t$$

$$\sigma_{[i]}^2(t) = \kappa_i + \alpha_i^{+} \cdot [u^2]_{[i]}^{[q]}(t)$$

  - ➤The quantities Z_t are assumed to be *N(0,1)*

MIXARCH for IBEX35

- The mixture model is

$$\text{Model 1} \quad \hat{X}_t = 0.0559\hat{X}_{t-1} + \sigma_t Z_t$$

$$\sigma_t^2 = 2.2194 + 0.1976\left(\hat{X}_{t-1} - 0.0559\hat{X}_{t-2}\right)^2$$

$$\text{Model 2} \quad \hat{X}_t = 0.1380\hat{X}_{t-1} + \sigma_t Z_t$$

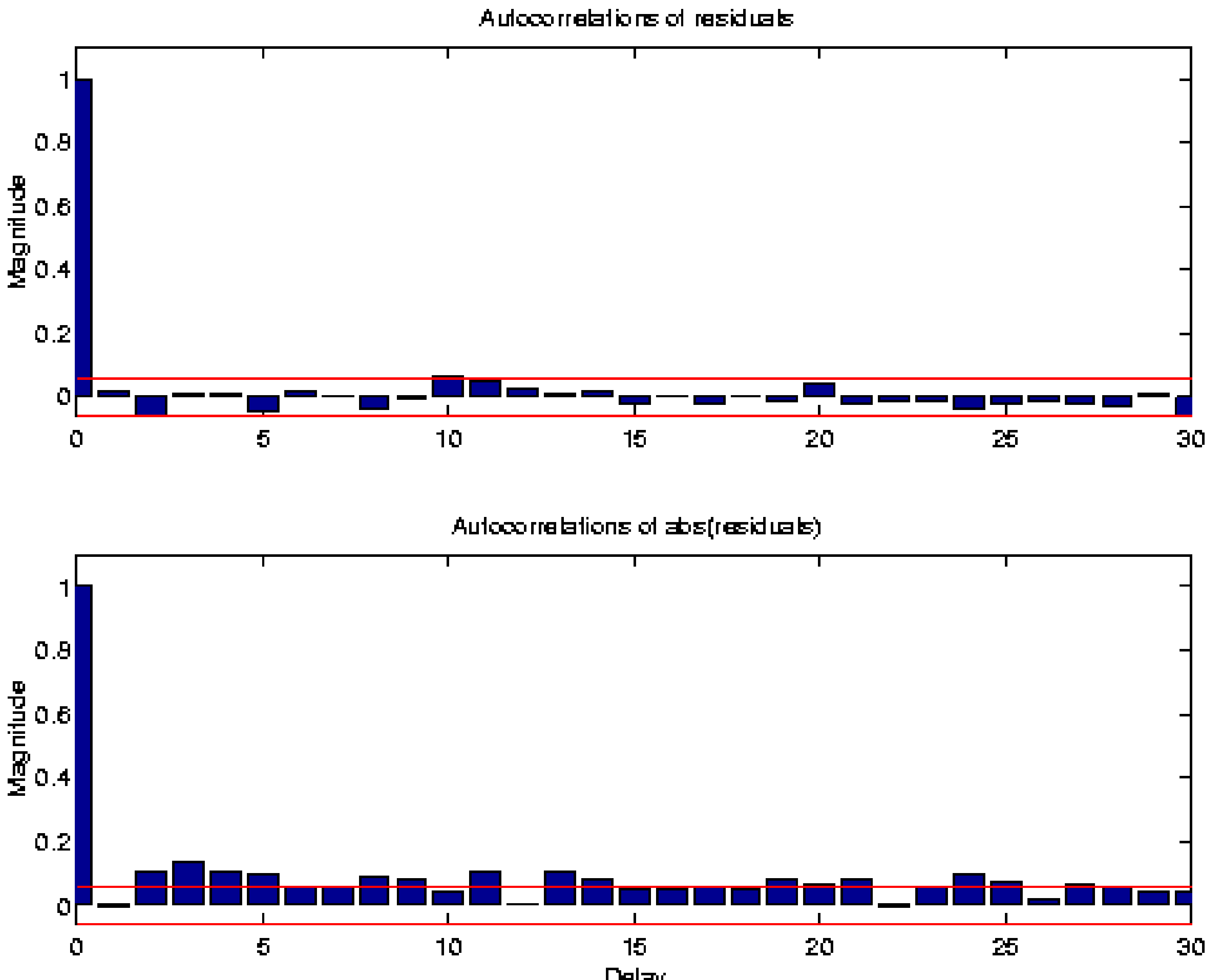
$$\sigma_t^2 = 0.6820 + 0.03821\left(\hat{X}_{t-1} - 0.1380\hat{X}_{t-2}\right)^2$$

- The probabilities for the mixture are

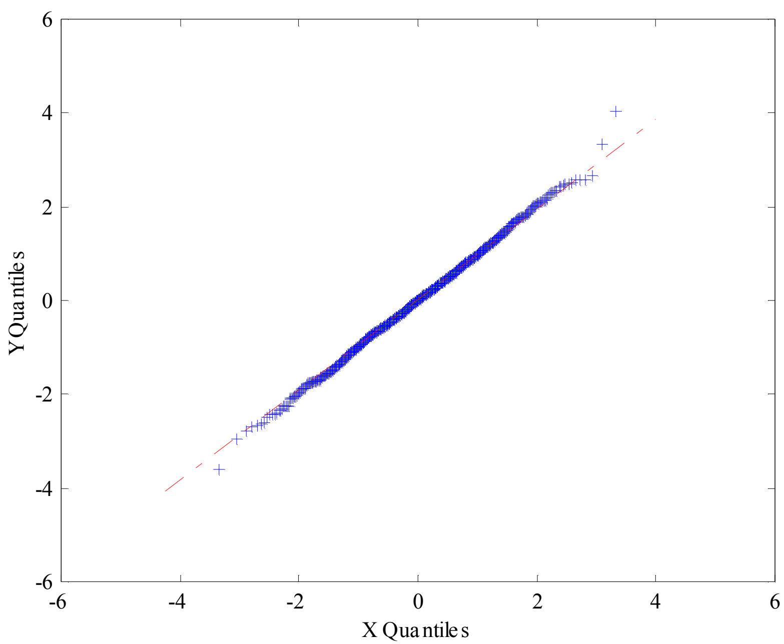
$$g_{[1]}(X_{t-1}) = \frac{1}{1 + \exp\{-0.6839(X_{t-1} - 2.5155)\}};$$

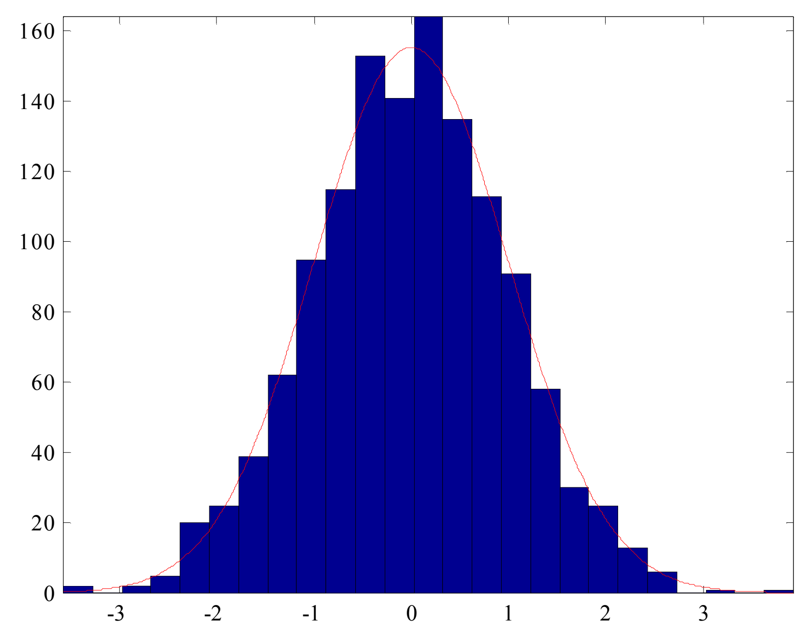
$$g_{[2]}(X_{t-1}) = 1 - g_{[1]}(X_{t-1})$$

Residual correlations: MIXARCH



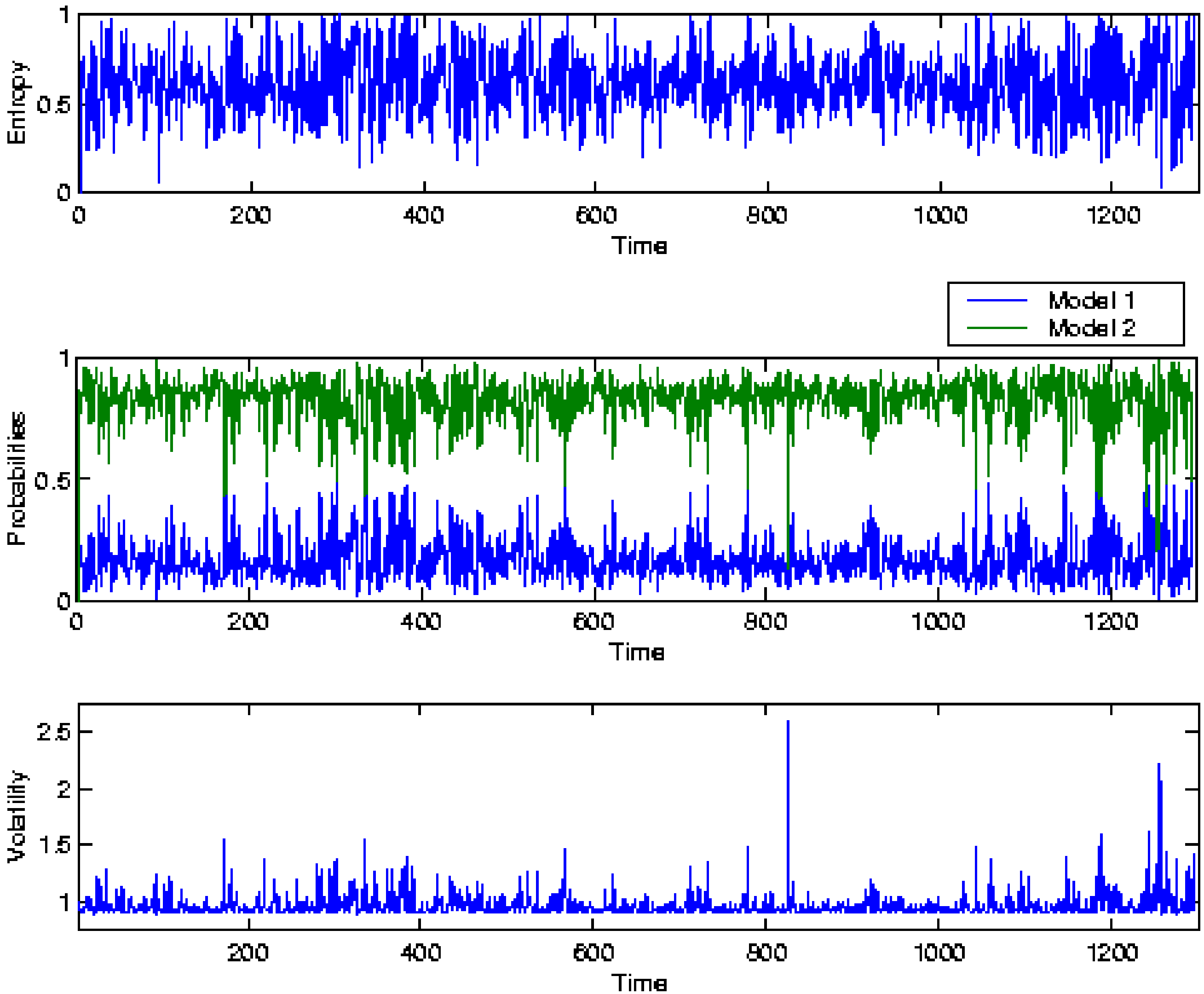
Normality hypothesis: MixARCH(1)





KS Test = 0.83

MIXARCH Model fit



Mixture of GARCH processes

- MixGARCH $\hat{X}_t = \phi_{[i]}^{+} \cdot \hat{\mathbf{X}}_t^{[m]} + u_{[i]}(t),$

  $$\textit{with probability} \;\; g_{[i]}(\mathbf{X}_t^{[r]}, \theta_{[i]})$$

  - The model for the residuals is

$$u_{[i]}(t) = \sigma_{[i]}(t) Z_t$$

$$\sigma_{[i]}^2(t) = \kappa_i + \alpha_i^{+} \cdot [u^2]_{[i]}^{[q]}(t) + \beta_i^{+} \cdot [\sigma^2]_{[i]}^{[p]}(t)$$

  - The quantities Z_t are assumed to be *N(0,1)*

MIXGARCH for IBEX35

- The mixture model is

$$\text{Model 1} \quad \hat{X}_t = 0.1255\hat{X}_{t-1} + \boldsymbol{\sigma}_t Z_t$$

$$\boldsymbol{\sigma}_t^2 = 0.0156 + 0.0778\left(\hat{X}_{t-1} - 0.1255X_{t-2}\right)^2 + 0.8937\boldsymbol{\sigma}_{t-1}^2$$

$$\text{Model 2} \quad \hat{X}_t = 0.3314\hat{X}_{t-1} + \boldsymbol{\sigma}_t Z_t$$

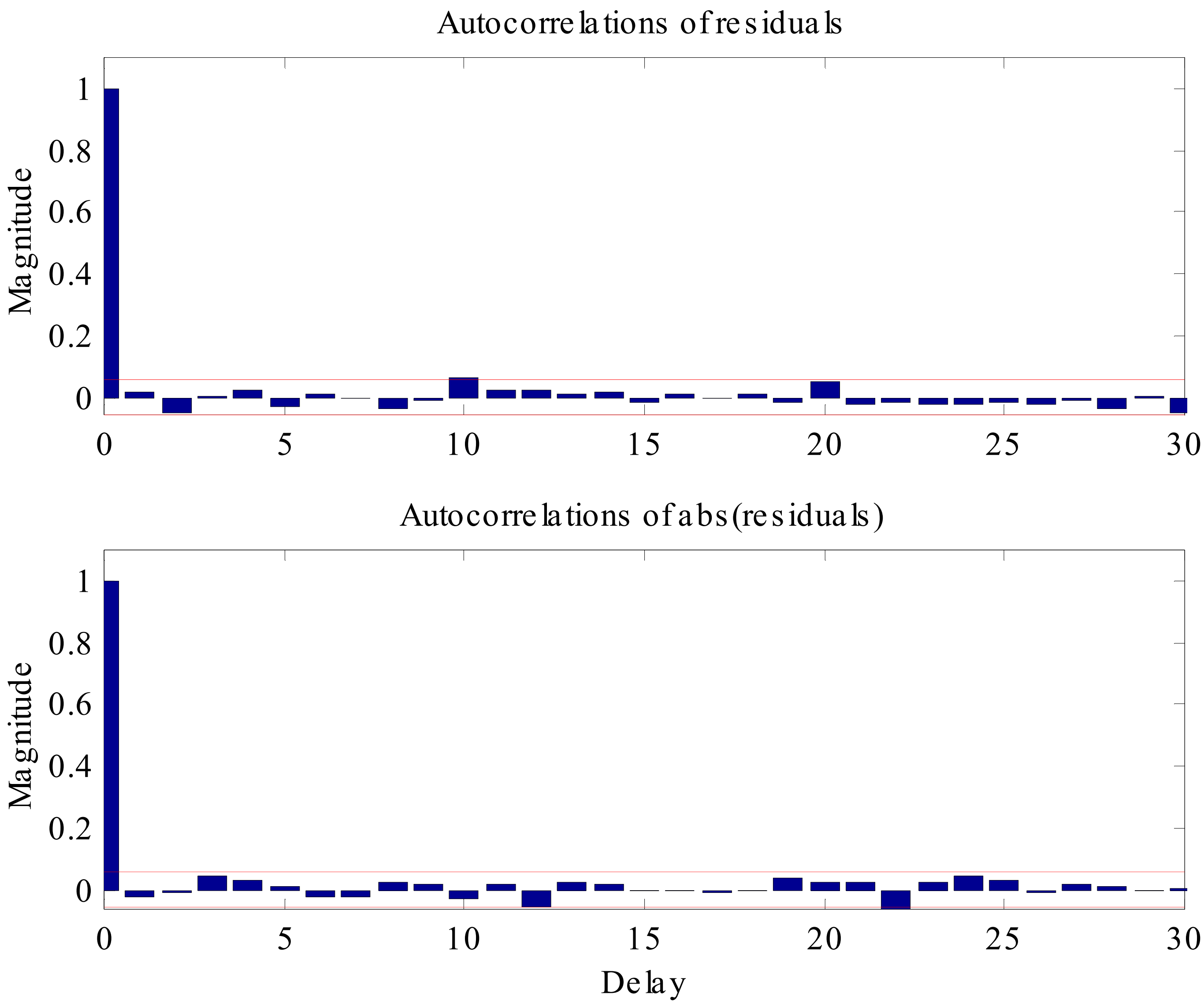
$$\boldsymbol{\sigma}_t^2 = 2.6230 + 0.0000\left(\hat{X}_{t-1} - 0.3314\hat{X}_{t-2}\right)^2 + 0.0285\boldsymbol{\sigma}_{t-1}^2$$

  - The probabilities for the mixture are

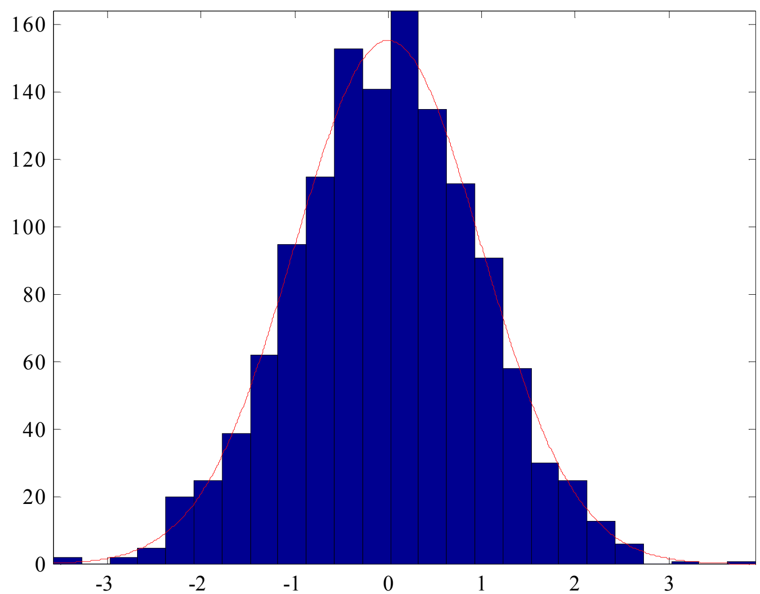
$$g_{[1]}(X_{t-1}) = \frac{1}{1 + \exp\left\{0.5418\,(\hat{X}_{t-1} - 4.8710)\right\}};$$

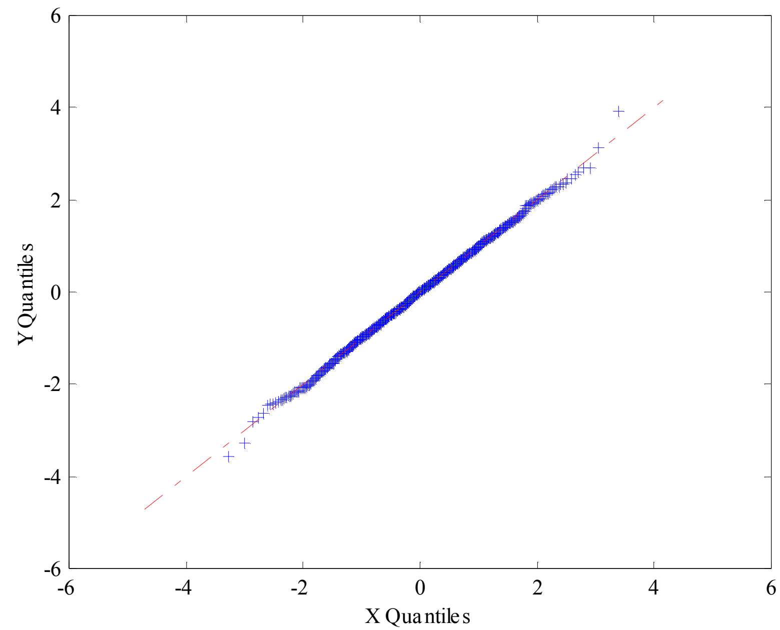
$$g_{[2]}(X_{t-1}) = 1 - g_{[1]}(X_{t-1})$$

Residual correlations: MIXGARCH



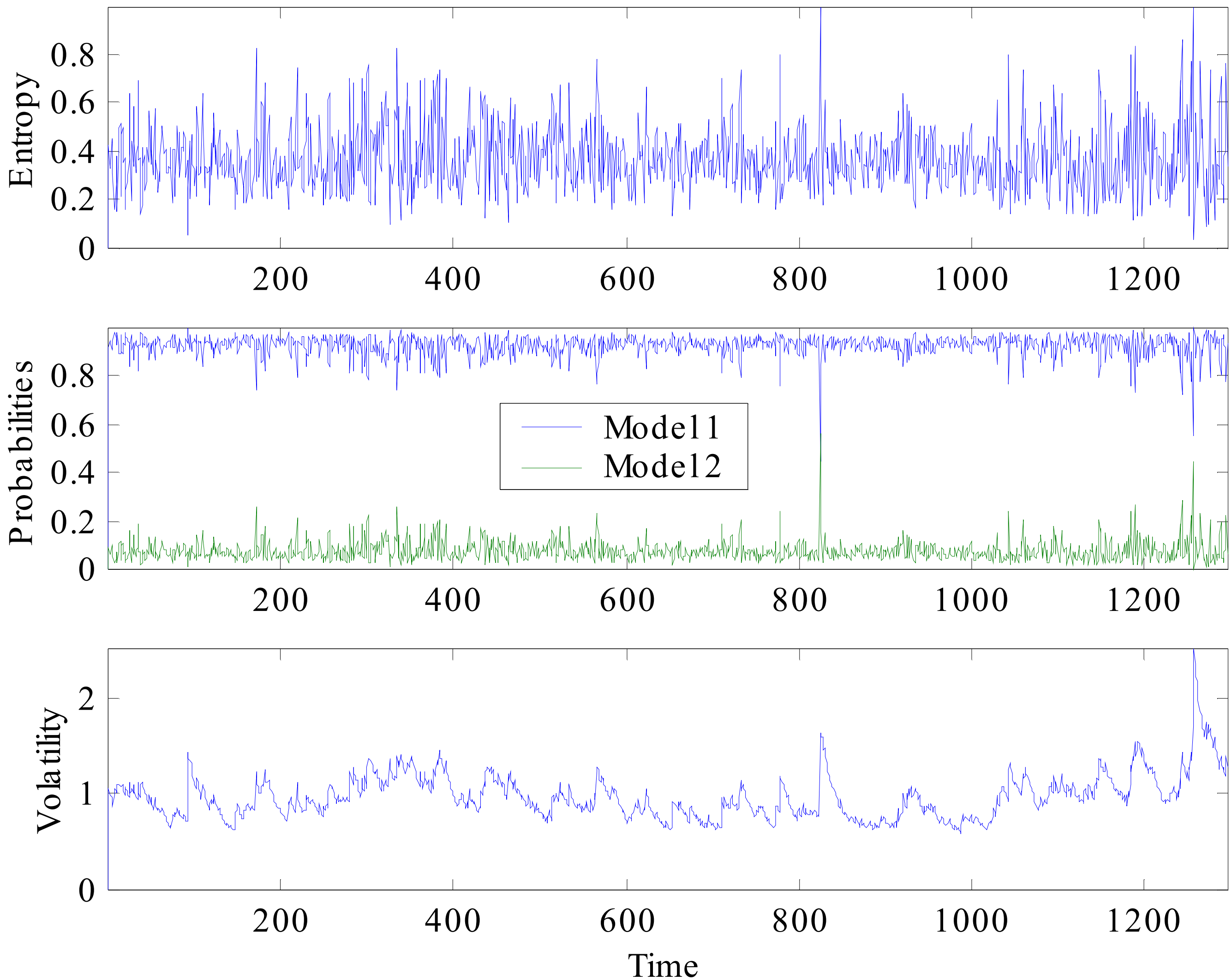
Normality hypothesis: MIXGARCH



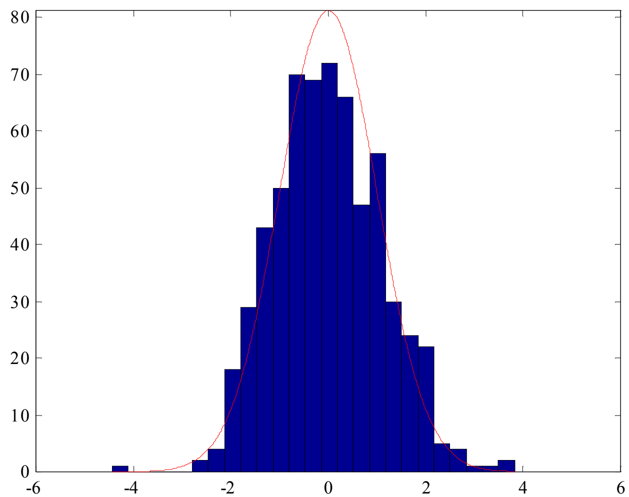


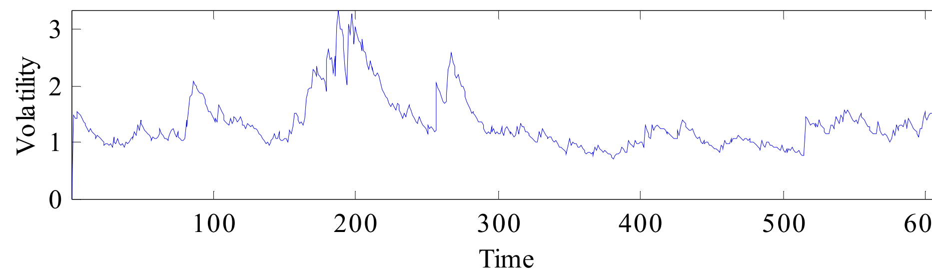
KS test = 0.95

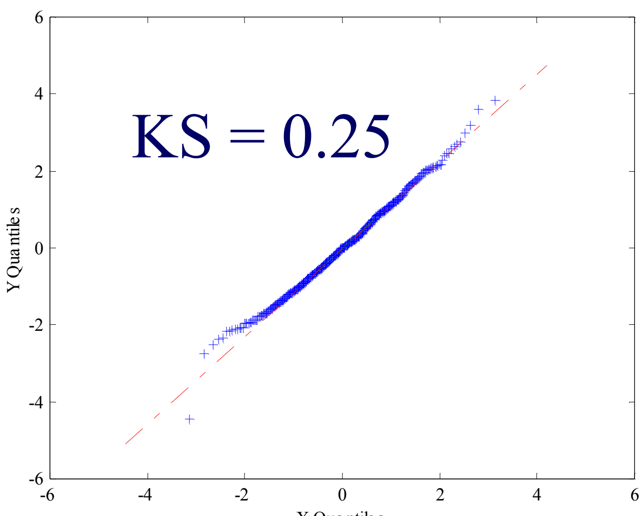
MIXGARCH Model fit

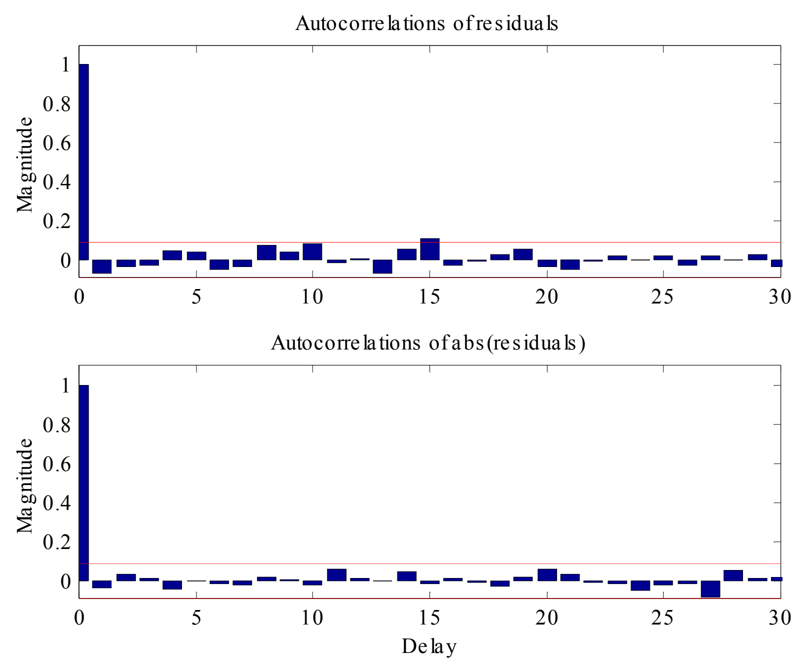


Test Data


80
70
60
50
40
30
20
10
0
-6
-4
-2
0
2
4
6


Volatility
3
2
1
0
100
200
300
400
500
600
Time


KS = 0.25
Y Quantiles
X Quantiles
6
4
2
0
-2
-4
-6

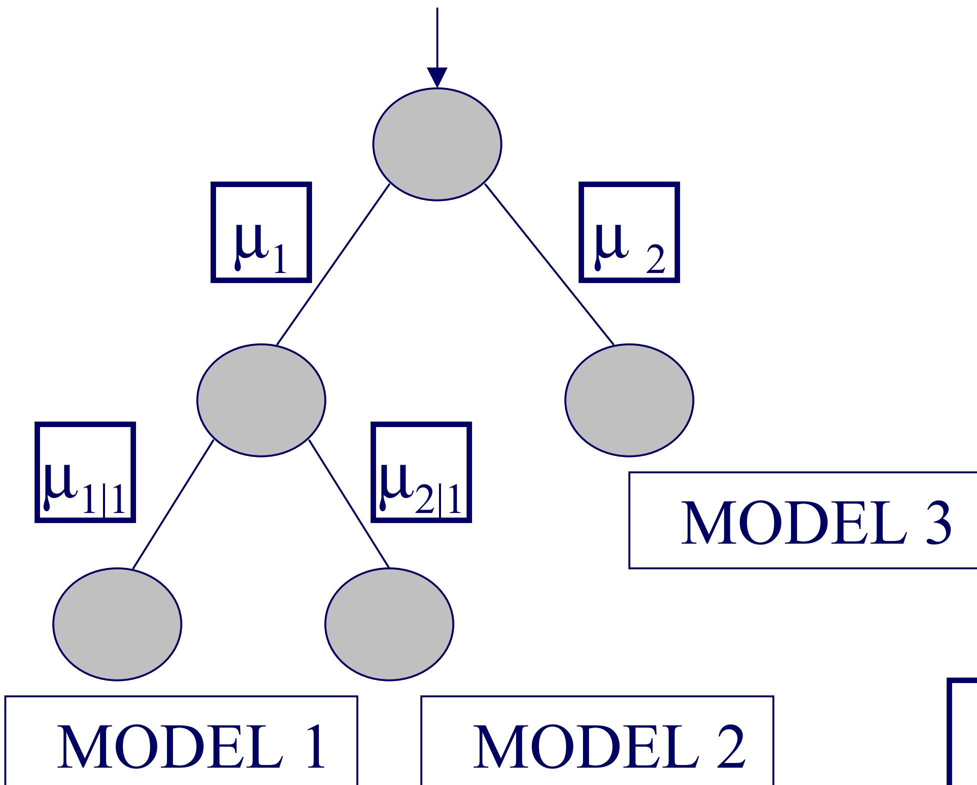

Autocorrelations of residuals
Magnitude
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30
Autocorrelations of abs(residuals)
Magnitude
Delay

Conclusions

- **Autoregressive mixture models** can be used to improve the modeling of financial time series
  - Model is piecewise-linear.
  - Memory effects.
  - Extreme events.
- **Difficulties:**
  - Parsimony? (Convex linear combination of simple base models)
  - Constrained optimization of LL function.
  - EM algorithm is not practical.

Perspectives:
Decision trees for TS analysis

Input = Vector of Delayed values



$$\mu_1 = \frac{\exp\left[b_1\left(\hat{X}_{t-1} + \sum_{k=1}^{r-1} a_{1k}\hat{X}_{t-k-1} - c_1\right)\right]}{1+\exp\left[b_1\left(\hat{X}_{t-1} + \sum_{k=1}^{r-1} a_{1k}\hat{X}_{t-k-1} - c_1\right)\right]}$$

$$\mu_2 = 1 - \mu_1$$

$$\mu_{1|1} = \frac{\exp\left[b_2\left(\hat{X}_{t-1} + \sum_{k=1}^{r-1} a_{2k}\hat{X}_{t-k-1} - c_2\right)\right]}{1+\exp\left[b_2\left(\hat{X}_{t-1} + \sum_{k=1}^{r-1} a_{2k}\hat{X}_{t-k-1} - c_2\right)\right]}$$

$$\mu_{2|1} = 1 - \mu_{1|1}$$

Model 1 $\mu_{11} = \mu_{1|1}\,\mu_1$;

Model 2 $\mu_{12} = \mu_{2|1}\,\mu_1$;

Model 3 μ_2

Acknowledgements

- **RiskLab Madrid.**
- **ETS Informática.**
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